Compact 3D Grid Drawings of Trees

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CONTRIBUTION

We show that perfect binary trees can be compactly embedded in 3D with optimal aspect ratio. A compact embedding is a straight-line drawing without crossings in which all points of a given grid except for maybe one are used.

Theorem: *The perfect binary tree with height $k = 3x - 1$ for $x \in \mathbb{N}$ with $n - 1 = 2^{k+1} - 1$ vertices has a compact embedding on the $\sqrt[3]{n} \times \sqrt[3]{n} \times \sqrt[3]{n}$ grid.*

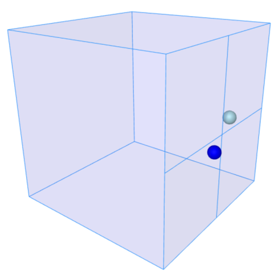
To show it, we adapt a recursive approach used by Akitaya *et al.* [1] for the 2D case: they recursively construct two types of compact embeddings of perfect binary trees on a square grid.

[1] H. A. Akitaya, M. Löffler, I. Parada. How to Fit a Tree in a Box. In: Proc. 26th International Symposium on Graph Drawing and Network Visualization (GD 2018). pp. 361–367. LNCS 11282, Springer, 2018.

We recursively construct three blocks F_k, G_k, and H_k which are compact embeddings of a perfect binary tree T_k of height k.

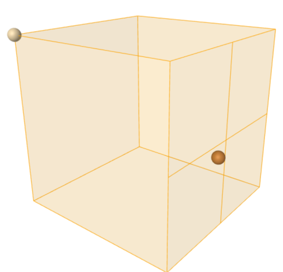
OVERVIEW OF THE BLOCKS

In all the blocks the **root** of T_k occupies the same position: $(2 \cdot 2^{(k-2)/3} - 1, 2^{(k-2)/3} - 1, 2^{(k-2)/3} - 1)$. The critical difference between the blocks is the placement of the empty node.



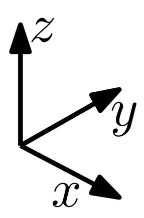
F_k Root node

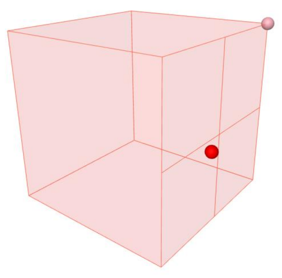
Empty node: $(2 \cdot 2^{\frac{k-2}{3}} - 1, 2^{\frac{k-2}{3}}, 2^{\frac{k-2}{3}})$



G_k Root node

Empty node: $(0, 0, 2 \cdot 2^{\frac{k-2}{3}} - 1)$





H_k Root node

Empty node: $(2 \cdot 2^{\frac{k-2}{3}} - 1, 2 \cdot 2^{\frac{k-2}{3}} - 1, 2 \cdot 2^{\frac{k-2}{3}} - 1)$

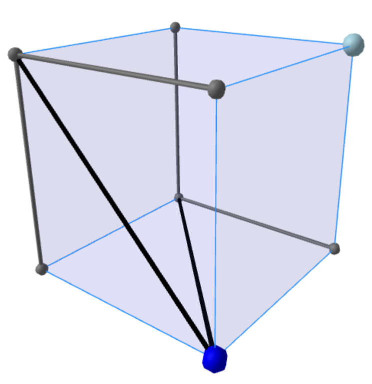
RECURSIVE DEFINITIONS

Each nontrivial block representing T_k can be recursively constructed from eight smaller sub-blocks representing T_{k-3}: four F_{k-3}, two G_{k-3}, and two H_{k-3}. Some need to be rotated as described later.

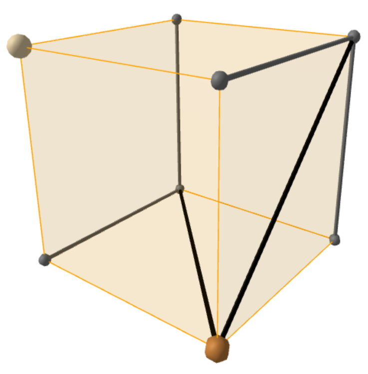
The sub-blocks are connected by non-intersecting edges using the root and empty nodes to complete the drawing of T_k. These connections (top three levels of T_k) are similar in all blocks:

- 3rd level: all root nodes ⟷ empty nodes of F_{k-3} sub-blocks (connecting to the same or else to an adjacent sub-block).
- 2nd level: these four empty nodes ⟷ empty nodes of the two H_{k-3} sub-blocks (connecting to a different yz plane).
- 1st level: these two nodes ⟷ empty node of a G_{k-3} sub-block.

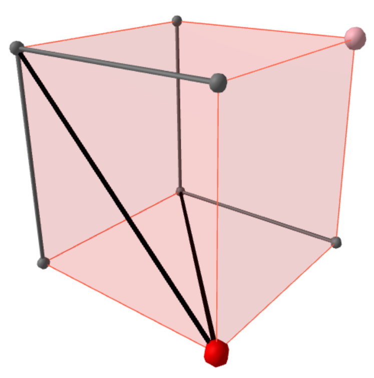
TRIVIAL BLOCKS



F_2



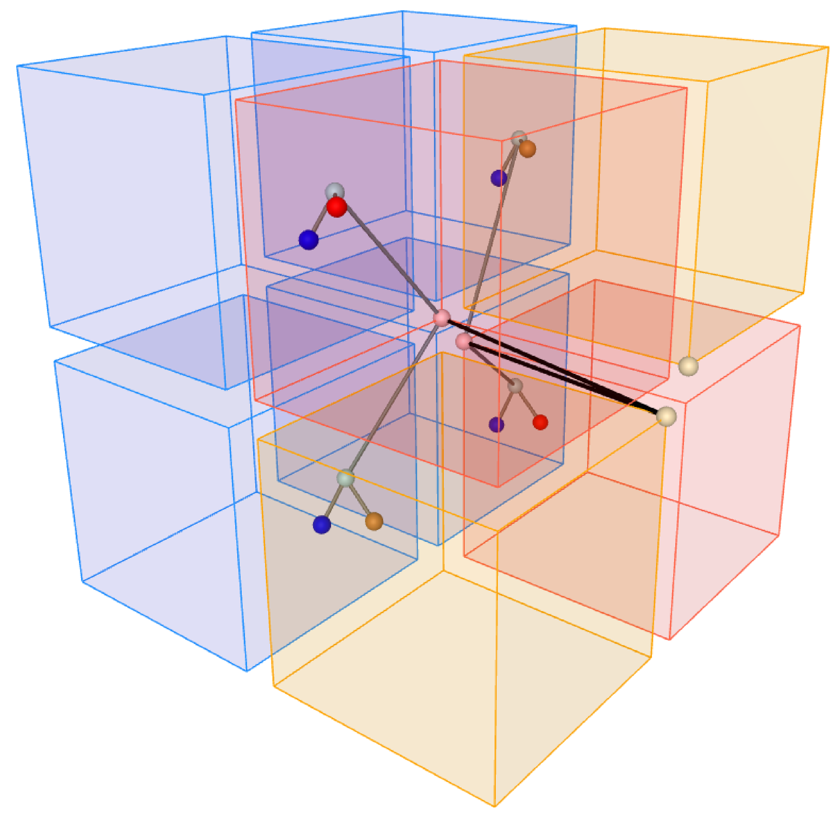
G_2



H_2

The eight sub-blocks are listed from bottom to top (-z to +z), from front to back (-y to +y), and from left to right (-x to +x).

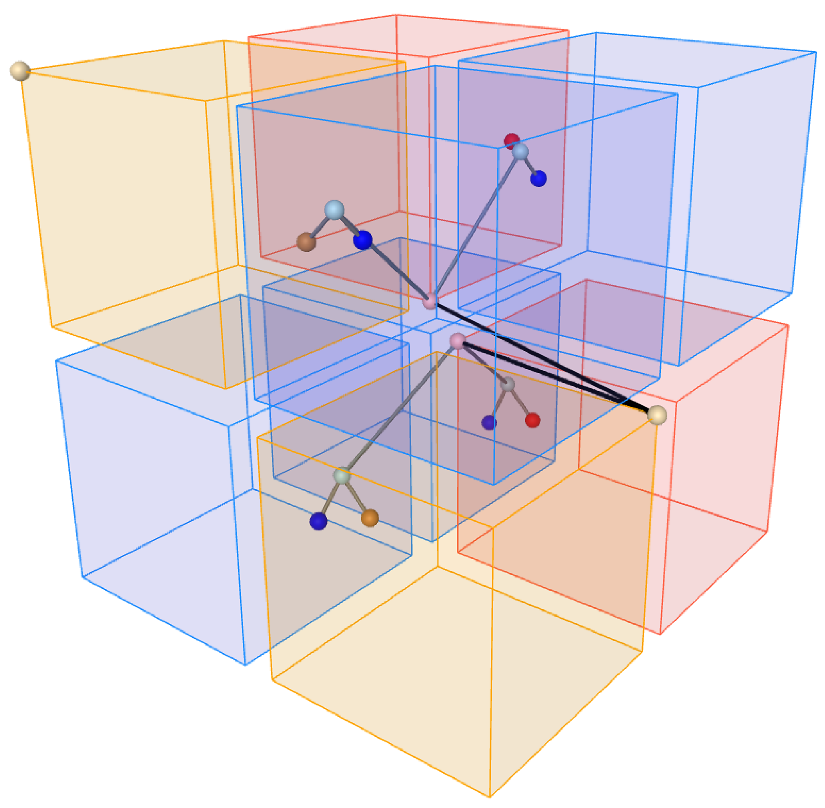
F_k BLOCK



Sub-blocks:

F_{k-3}, G_{k-3} (rot. $\pi|_z$), F_{k-3}, H_{k-3} (rot. $\pi|_z$), F_{k-3}, H_{k-3} (rot. $\pi|_z$ and rot. $\pi|_x$), F_{k-3}, and G_{k-3} (rot. $\pi|_z$ and rot. $\pi|_x$).

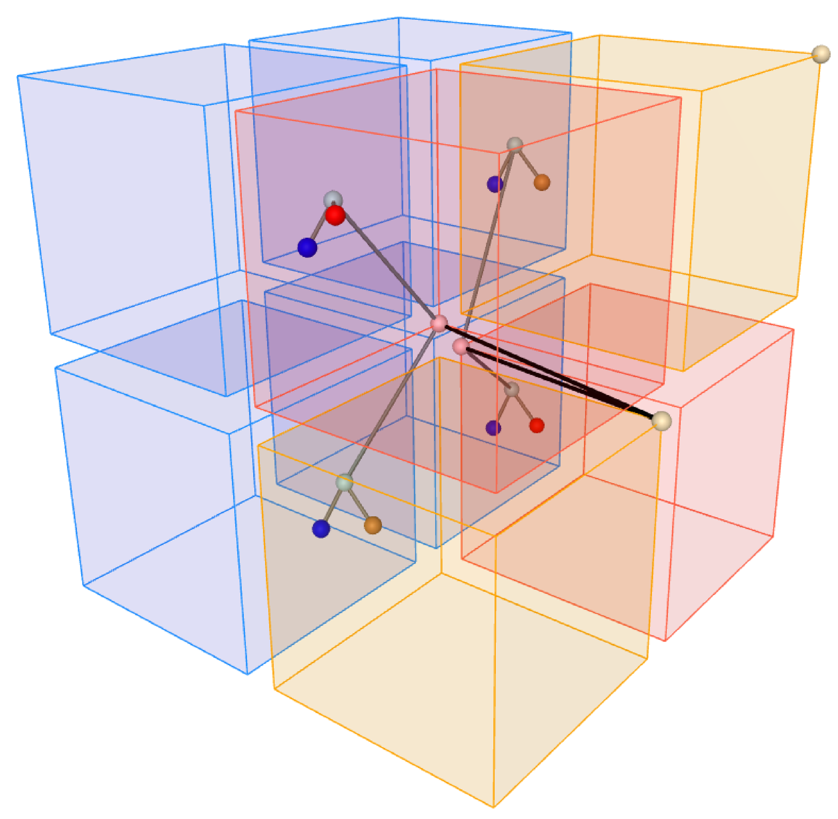
G_k BLOCK



Sub-blocks:

F_{k-3}, G_{k-3} (rot. $\pi|_z$), F_{k-3}, H_{k-3} (rot. $\pi|_z$), G_{k-3}, F_{k-3} (rot. $\pi|_z$), H_{k-3} (rot. $\pi|_x$), and F_{k-3} (rot. $\pi|_z$).

H_k BLOCK



Sub-blocks:

F_{k-3}, G_{k-3} (rot. $\pi|_z$), F_{k-3}, H_{k-3} (rot. $\pi|_z$), F_{k-3}, H_{k-3} (rot. $\pi|_z$ and rot. $\pi|_x$), F_{k-3}, and G_{k-3} (rot. $\pi|_z$).

OPEN QUESTION

Which other trees admit a compact embedding in 2D or 3D (with/without optimal aspect ratio)?

Scan the QR codes for interactive 3D drawings