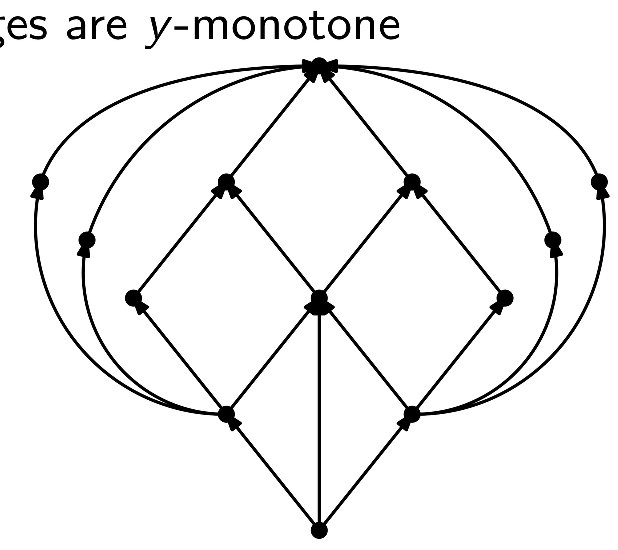
Bitonicity and Upwardness

Splits, Bends and their Relation to Area Requirements

Patrizio Angelini[1] Michael A. Bekos[2] Henry Förster[2] Martin Gronemann[3]

Upward Planar Drawings

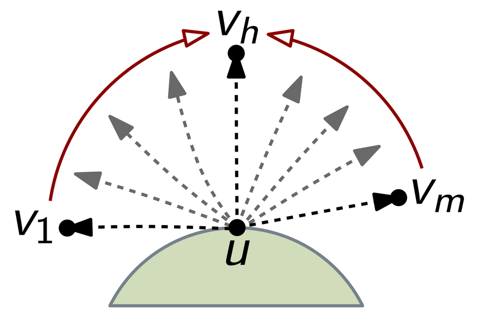
Def: Edges are y-monotone



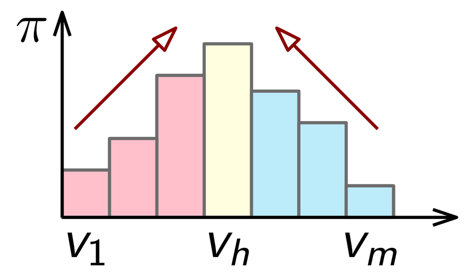
Di Battista, Tamassia, Tollis 1992 [3]
There are st-planar graphs that require exponential area in every straight-line upward planar drawing.

Bitonic st-orderings

Given an st-ordering $\pi : V \mapsto \{1, \dots, |V|\}$



$S(u) = \{v_1, \dots, v_m\}$
Successor list of u ordered as in the embedding

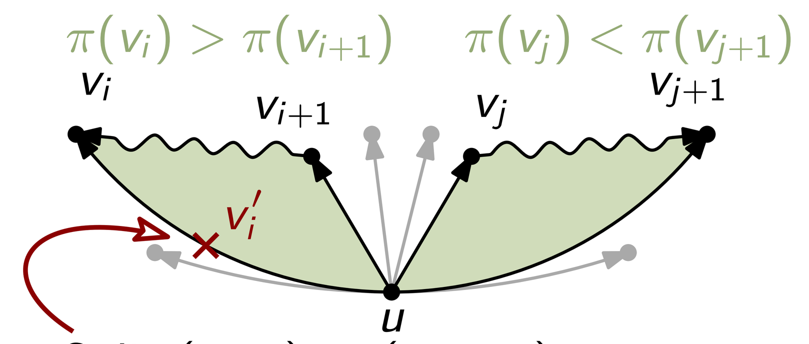


$\pi(v_1) < \dots < \pi(v_h) > \dots > \pi(v_m)$
increasing and then decreasing
$\Rightarrow$ **bitonic sequence**

$\forall u \in V : S(u)$ is bitonic w.r.t. $\pi \Rightarrow \pi$ is a **bitonic** st-ordering

Edge Splitting

No bitonic st-ordering $\Rightarrow$ Forbidden configuration



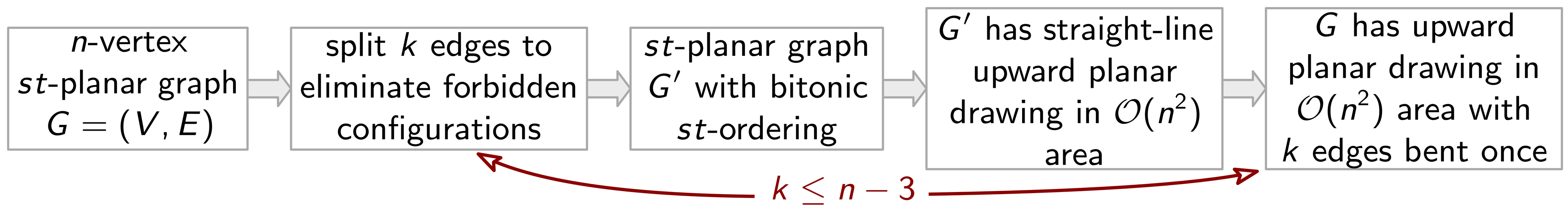
Split (u, v_i) or (u, v_{j+1})
$\Rightarrow$ No path from v_{i+1} to v_i'

Forbidden configuration eliminated!

Research Questions

1. **How many bends are required for polynomial area?**
2. **Is the bitonic st-ordering approach producing good results?**

Upward Planar Drawings via Bitonic st-Orderings [5]



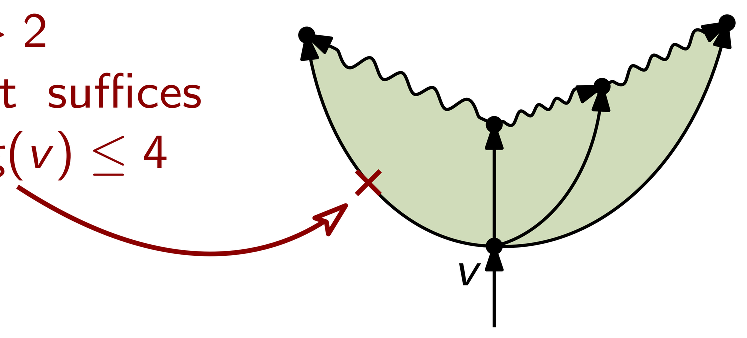
$\exists$ graphs that require $n - 3$ splits but have a linear area straight-line upward drawing

Bends vs. Splits

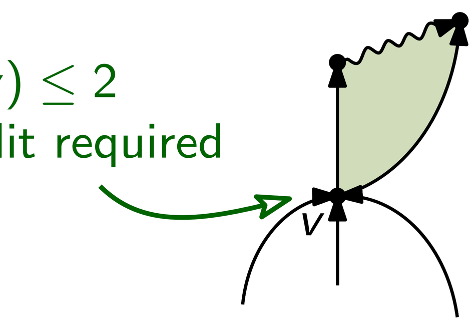
maximum degree	bends for polynomial area	splits for bitonicity
3	0 [2]	0 [2]
4	**$\geq$ n/2-O(log n)**	**$\leq$ n/2**
5	**$\geq$ n/2-O(log n)**	**$\leq$ n/2**
≥ 6	**$\geq$ n-O(log n)**	$\leq$ n-3 [5]

Maximum Degree 5 $\Rightarrow$ n/2 Splits

outdeg(v) > 2
$\Rightarrow$ one split suffices since outdeg(v) ≤ 4



outdeg(v) ≤ 2
$\Rightarrow$ no split required



outdeg(v) ≤ 2 in G or its reverse orientation $\tilde{G}$

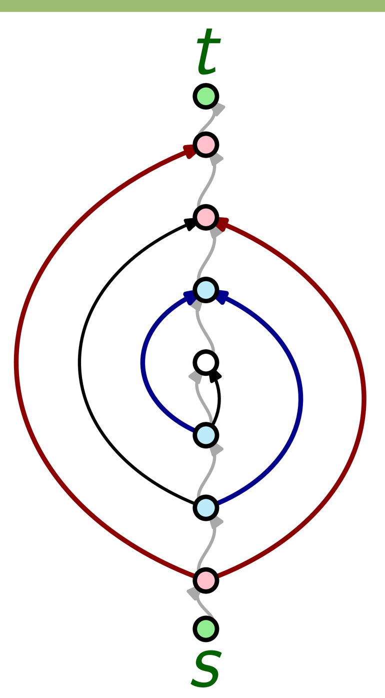
$\Rightarrow$ $n/2$ splits suffice for G or $\tilde{G}$

k-Coil [3,4]

k edges spiraling around an st-path

alternating sequence of V- and Λ-shapes

requires $\Omega(2^k)$ area in any upward planar straight-line drawing [4]



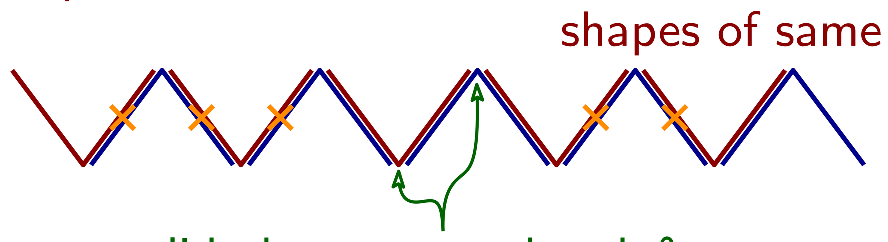
Coil Lemma

Lemma: *A graph containing a k-coil ξ has no polynomial area upward planar drawing in which less than $k/2 - O(\log k)$ edges of ξ are bent.*

1. Sequence of b bent edges along ξ invalidates at most $2b$ shapes

   b consecutive edges bent
   $\Rightarrow$ $b + 1$ shapes have a bend

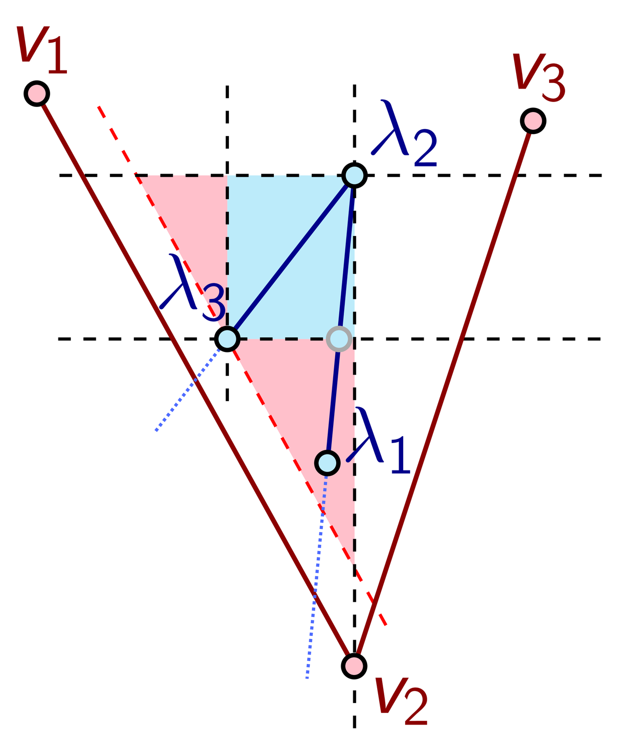
   b is even
   $\Rightarrow$ two consecutive bendless shapes of same type

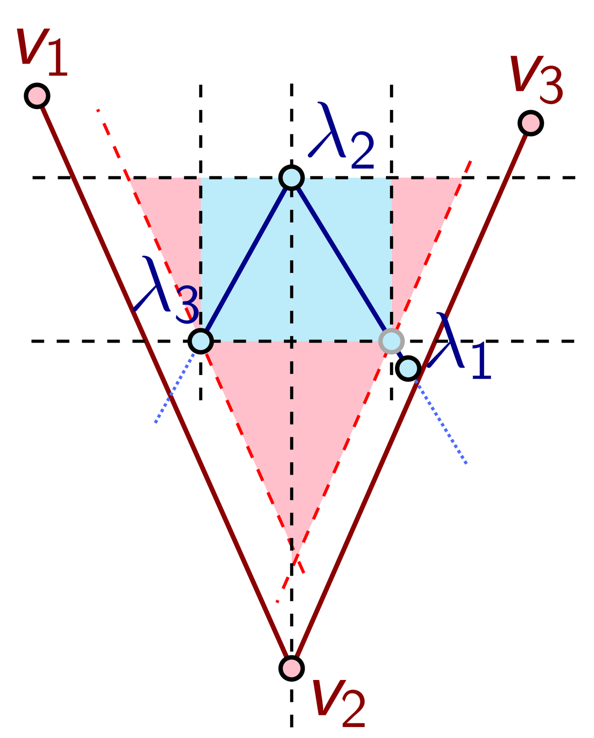
   

   valid shape = no bend & next bendless shape is of different type

2. Assume less than $k/2 - O(\log k)$ bent edges
   $\Rightarrow \exists$ alternating sequence of $c = \omega(\log k)$ valid shapes

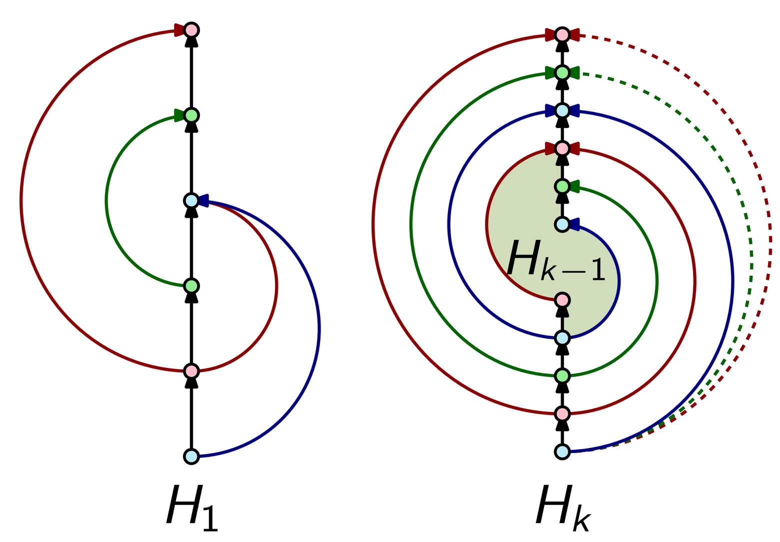
3. A valid V-shape nests the next valid Λ-shape
   $\Rightarrow$ via shearing, we can reduce to the following two cases





$\Rightarrow \omega(4^c) = \omega(poly(n))$ area is required ↯

Maximum Degree 4

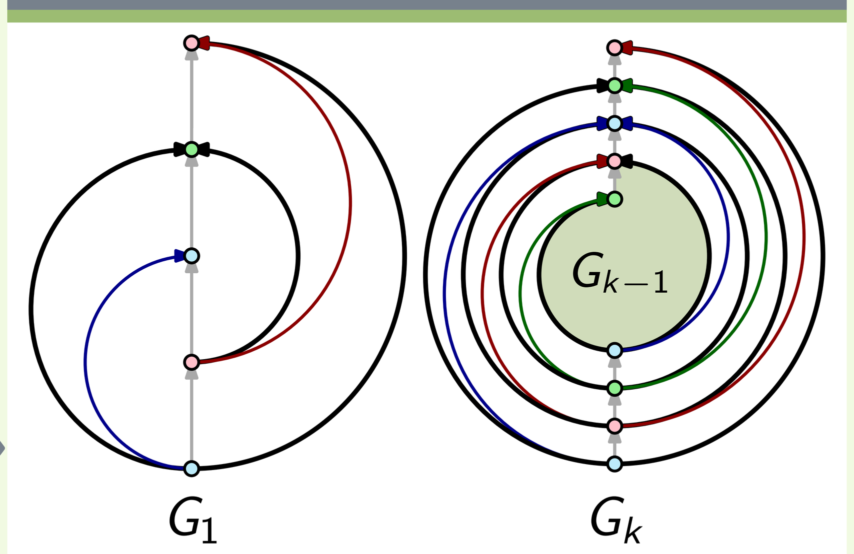


$\Rightarrow$ $n/2 - 2$ splits in H_k and $\tilde{H}_k$

$\Rightarrow$ contains three $n/3 - O(1)$-coils

$\Rightarrow$ requires $n/2 - O(\log n)$ bends for polynomial area

Higher Degree



$\Rightarrow$ $n - 5$ splits in G_k and $\tilde{G}_k$ [1]

$\Rightarrow$ contains a $n - O(1)$-coils and three $n/3 - O(1)$-coils

$\Rightarrow$ requires $n - O(\log n)$ bends for polynomial area

References

[1] P. Angelini, M. Bekos, H. Förster, M. Gronemann. Bitonic st-Orderings for Upward Planar Graphs: The Variable Embedding Setting. WG 2020: 339-351

[2] M. Bekos, E. Di Giacomo, W. Didimo, G. Liotta, F. Montecchiani. Universal Slope Sets for Upward Planar Drawings. Graph Drawing 2018: 77-91

[3] G. Di Battista, R. Tamassia, and I. G. Tollis. Area requirement and symmetry display of planar upward drawings. Discret. Comp. Geom., 7:381–401, 1992.

[4] F. Frati. On Minimum Area Planar Upward Drawings of Directed Trees and Other Families of Directed Acyclic Graphs. Int. J. Comp. Geom. Appl. 18(3): 251-271 (2008)

[5] M. Gronemann. Bitonic st-orderings for Upward Planar Graphs. Graph Drawing 2016: 222-235

[1] John Cabot University Rome [2] Wilhelm-Schickard-Institut für Informatik, Universität Tübingen [3] Theoretical Computer Science, Osnabrück University