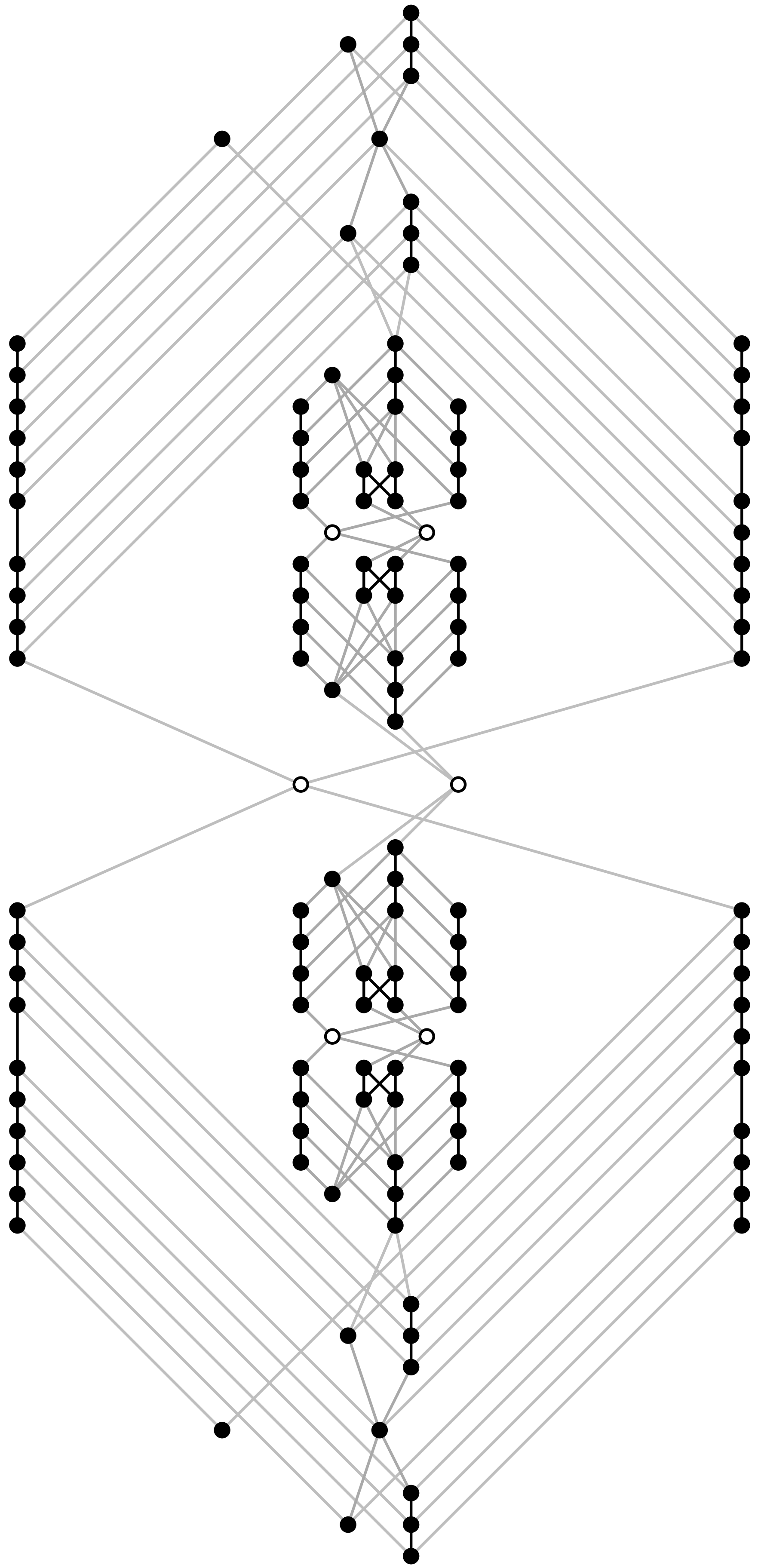


On the Queue-Number of Partial Orders

Stefan Felsner

Technische Universität Berlin

Torsten Ueckerdt

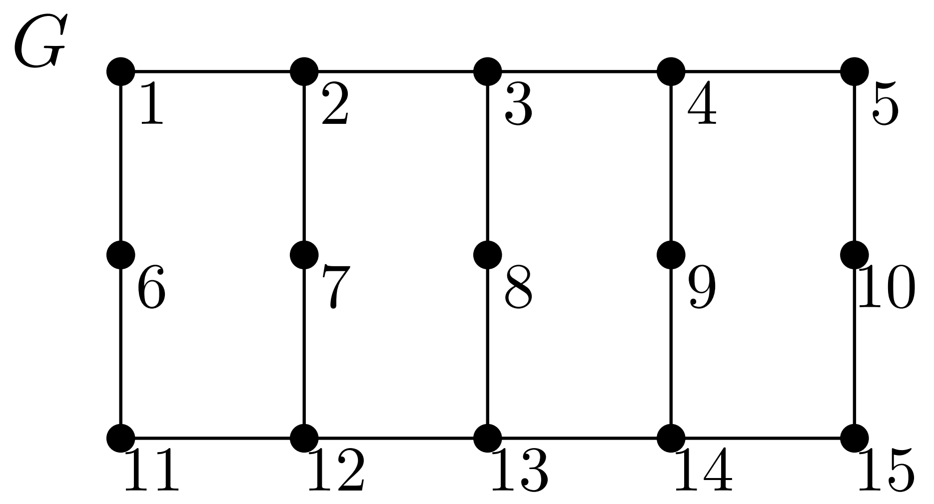
Karlsruhe Institute of Technology

Kaja Wille

Technische Universität Berlin

29th International Symposium on Graph Drawing and Network Visualization

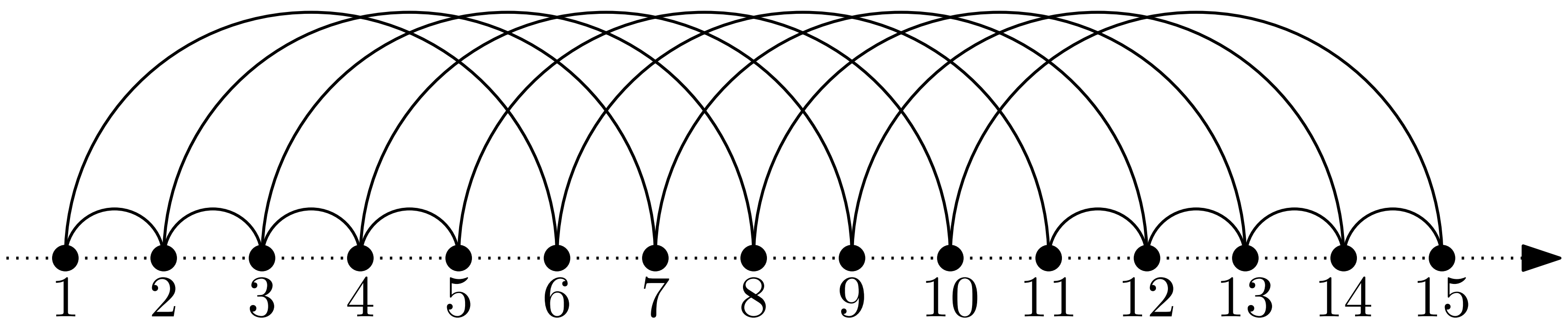
Tübingen, September 16



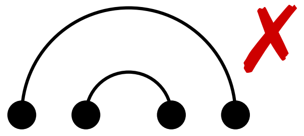
$\rightsquigarrow$

queue layout

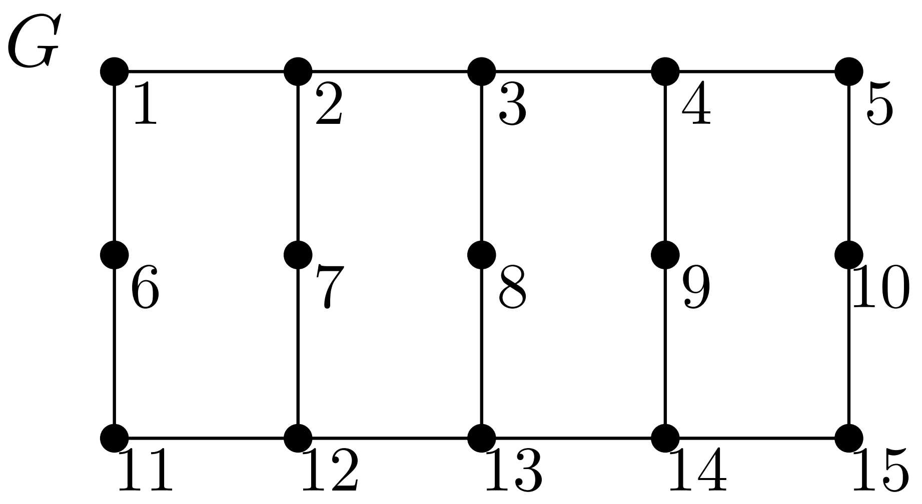
$\mathrm{qn}(G) \leq 2$



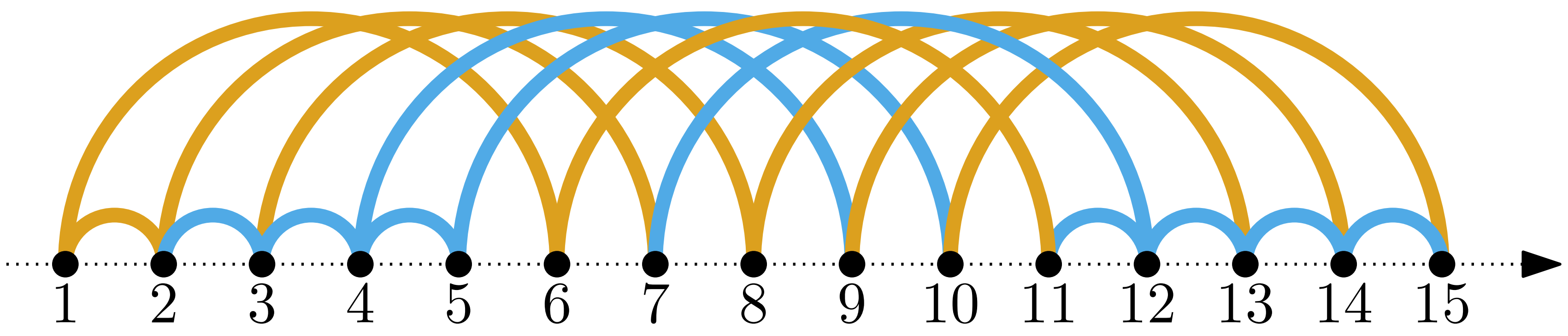
▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

$\mathrm{qn}(G) =$ min k s.t. $\left\{ \begin{array}{l} \exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition} \end{array} \right\}$ with  in each part

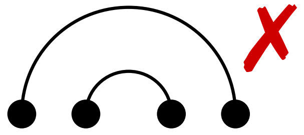
no nesting

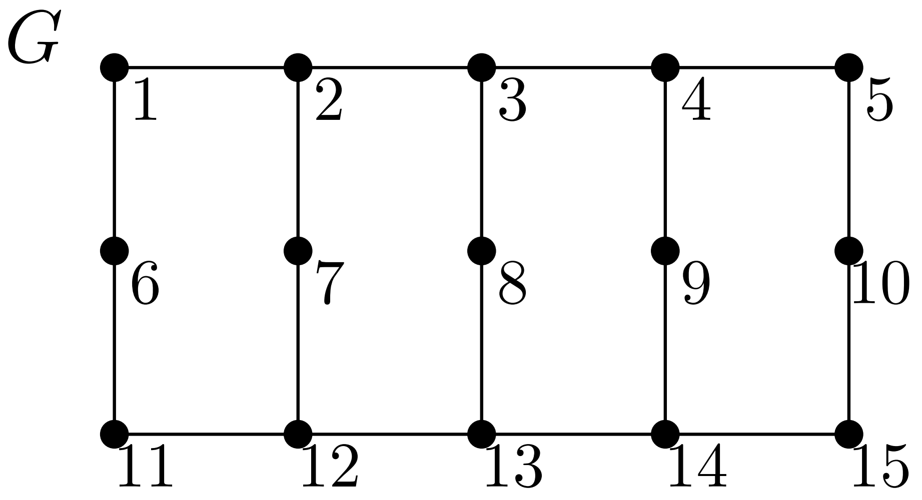


$\rightsquigarrow$

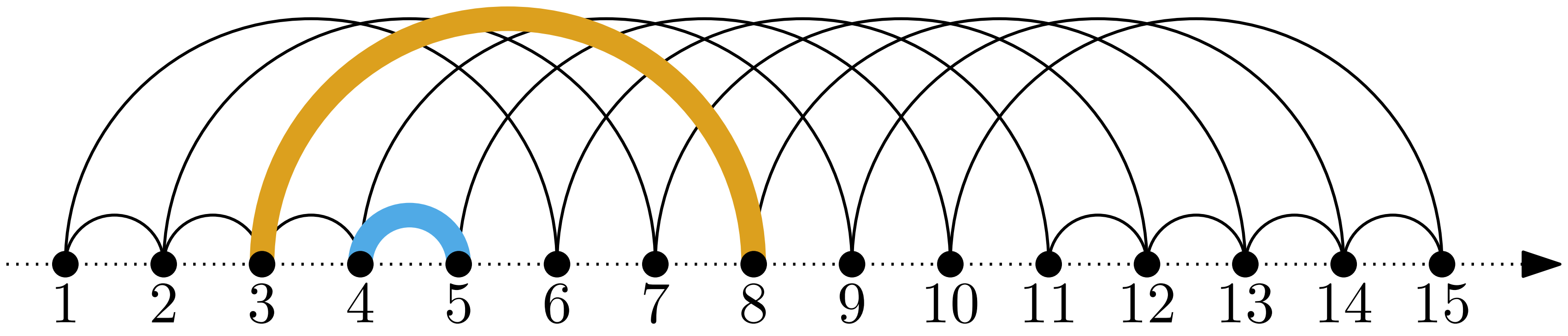


▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992)**.**

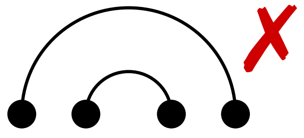
$\mathrm{qn}(G)$ = min k s.t. $\left\{\begin{array}{l}\exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition}\end{array}\right\}$ with  in each part



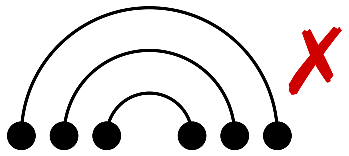
$\rightsquigarrow$



▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

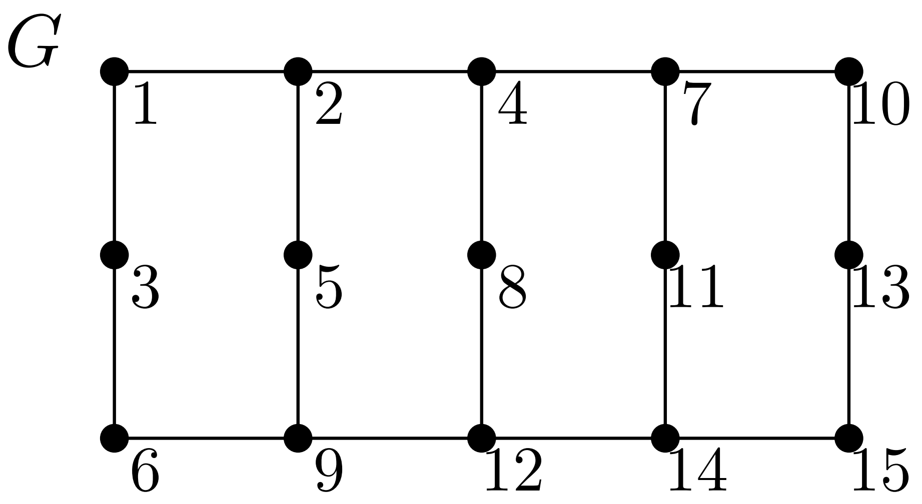
$\mathrm{qn}(G) =$ min k s.t. $\left\{\begin{array}{l}\exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition}\end{array}\right\}$ with  in each part

no nesting

$\mathrm{qn}(G) =$ min k s.t. $\exists$ vertex ordering with 

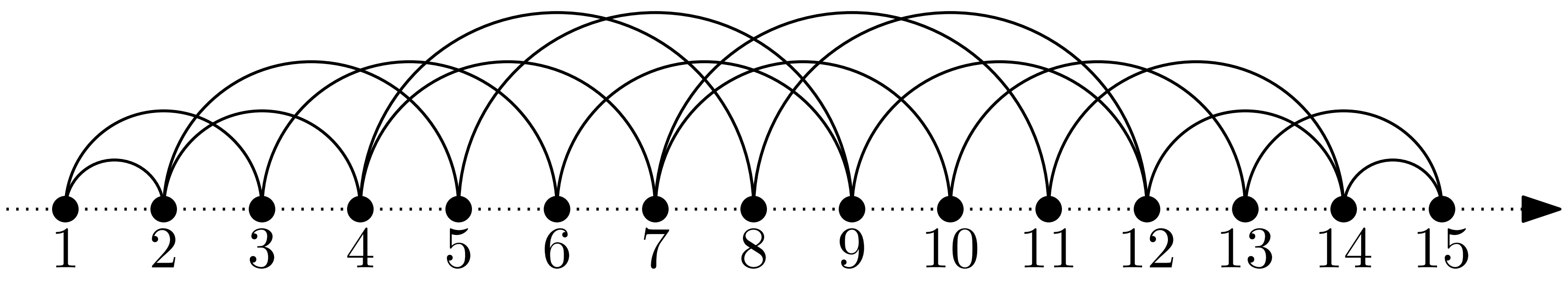
no $(k+1)$-nesting*

*also called $(k+1)$-rainbow

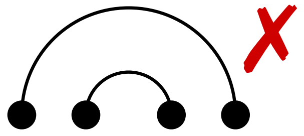


$\rightsquigarrow$

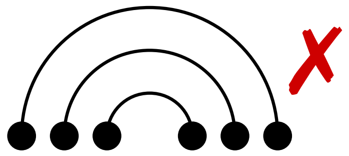
queue layout $\quad\quad$ $\mathrm{qn}(G) = 1$



▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

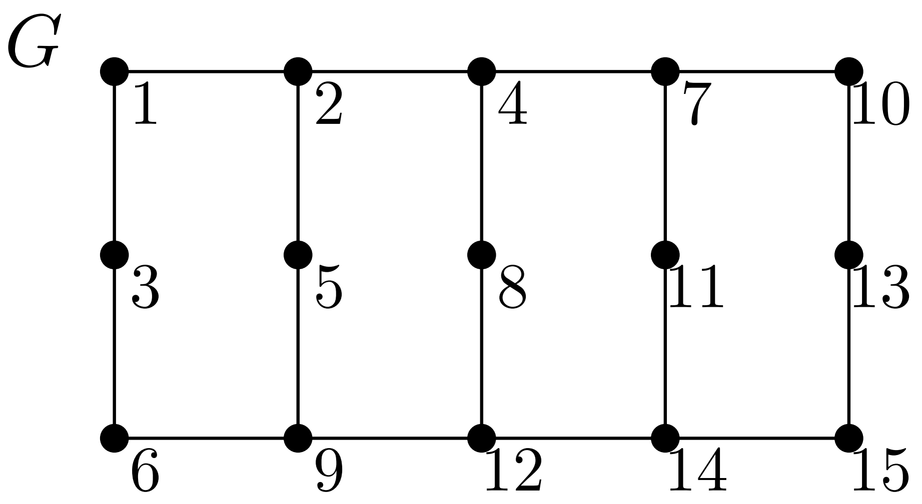
$\mathrm{qn}(G) = \min k$ s.t. {∃ vertex ordering / ∃ k-edge partition} with  in each part

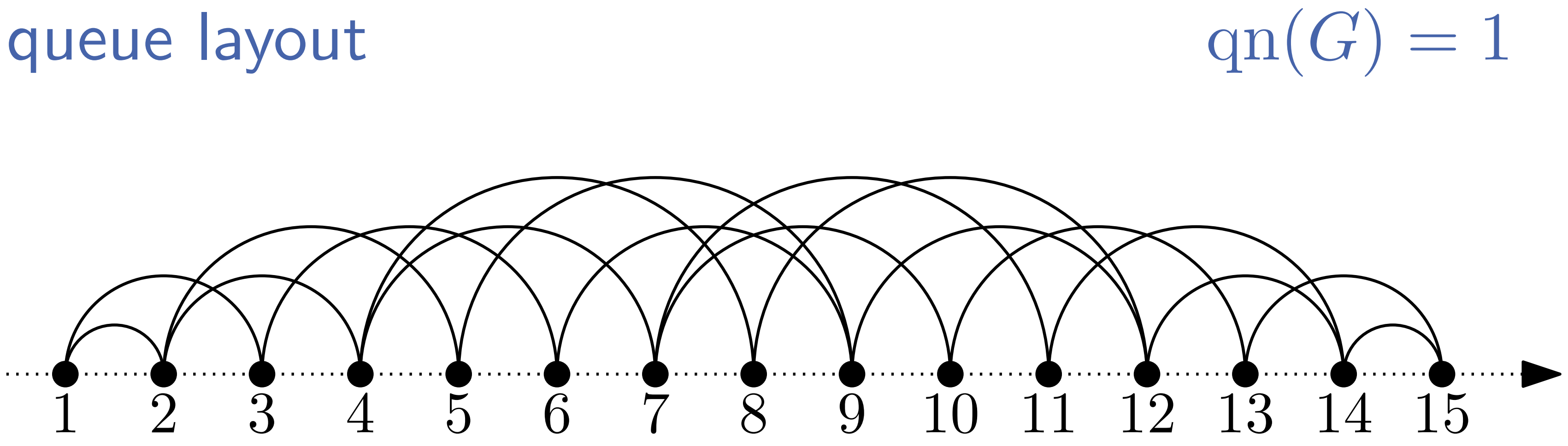
no nesting

$\mathrm{qn}(G) = \min k$ s.t. ∃ vertex ordering with 

no $(k+1)$-nesting*

*also called $(k+1)$-rainbow



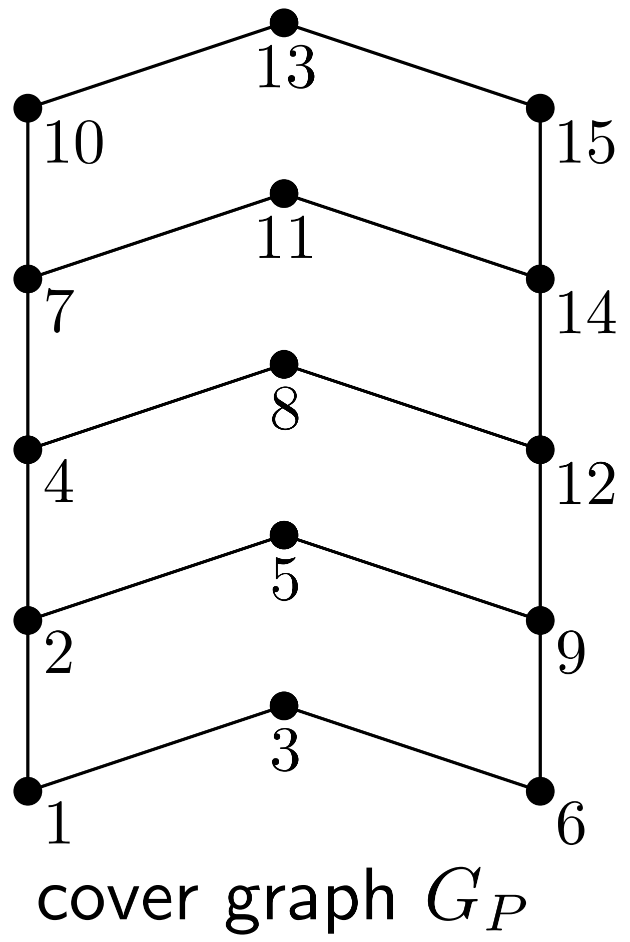


$\mathrm{qn}(G) = 1$

▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

$\mathrm{qn}(G) =$ min k s.t. {∃ vertex ordering, ∃ k-edge partition} with [no nesting] in each part

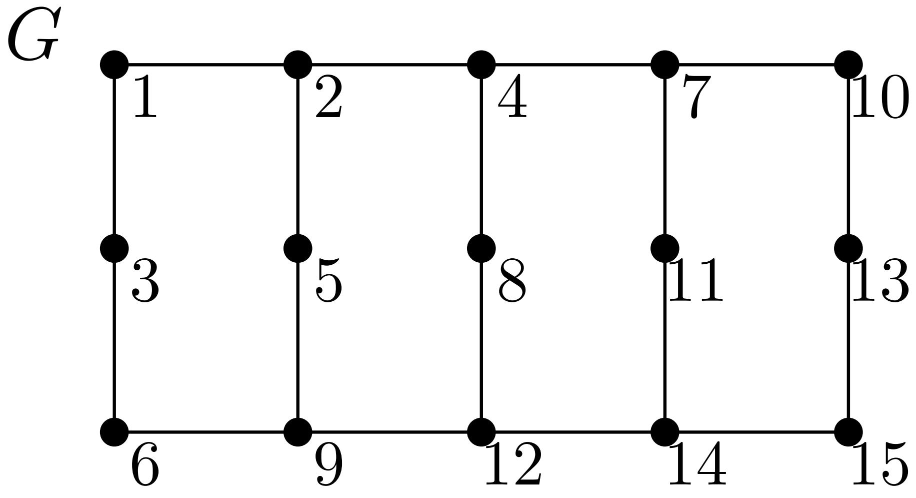
$\mathrm{qn}(G) =$ min k s.t. ∃ vertex ordering with no $(k+1)$-nesting*



▷ **Queue-Number of a Poset** (Heath, Pemmaraju 1997).

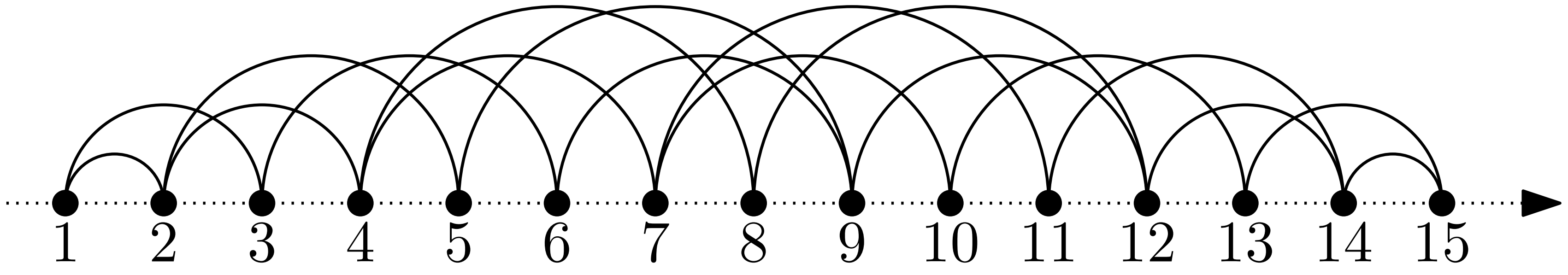
$\mathrm{qn}(P) =$ min k s.t. ∃ linear extension with no $(k+1)$-nesting of covers

*also called $(k+1)$-rainbow



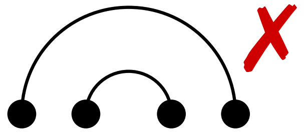
⇝

queue layout $\mathrm{qn}(G) = 1$



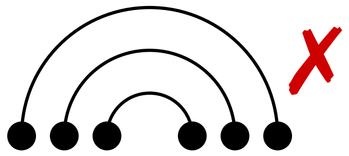
▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

$\mathrm{qn}(G) =$ min k s.t. { ∃ vertex ordering / ∃ k-edge partition } with



no nesting in each part

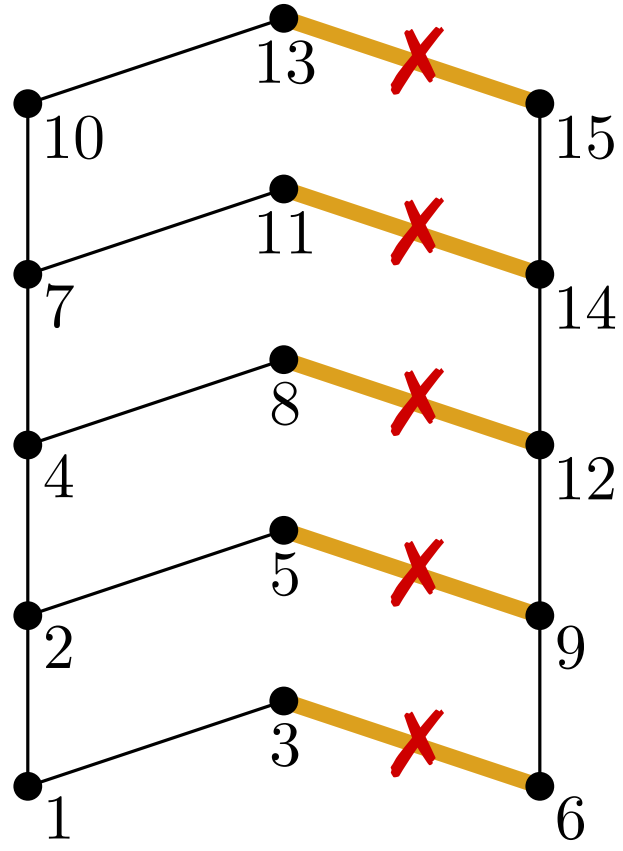
$\mathrm{qn}(G) =$ min k s.t. ∃ vertex ordering with



no $(k+1)$-nesting*

▷ **Queue-Number of a Poset** (Heath, Pemmaraju 1997).

$\mathrm{qn}(P) =$ min k s.t. ∃ linear extension with no $(k+1)$-nesting of covers

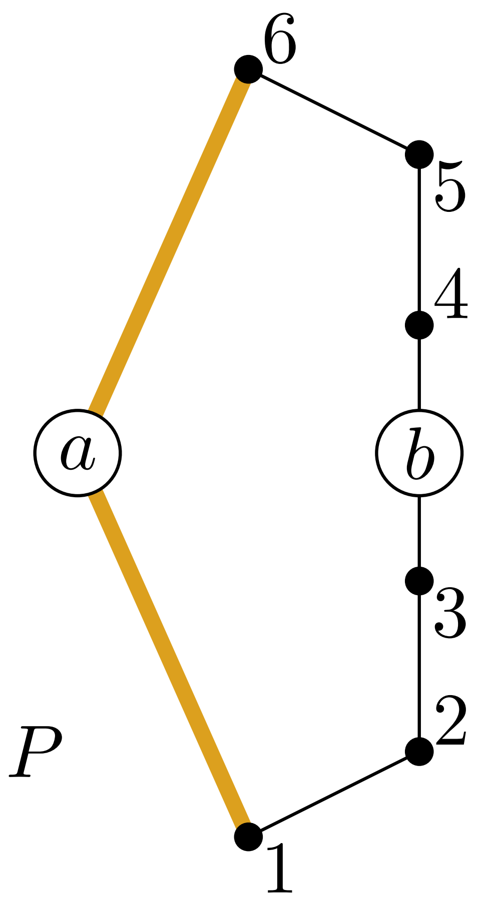


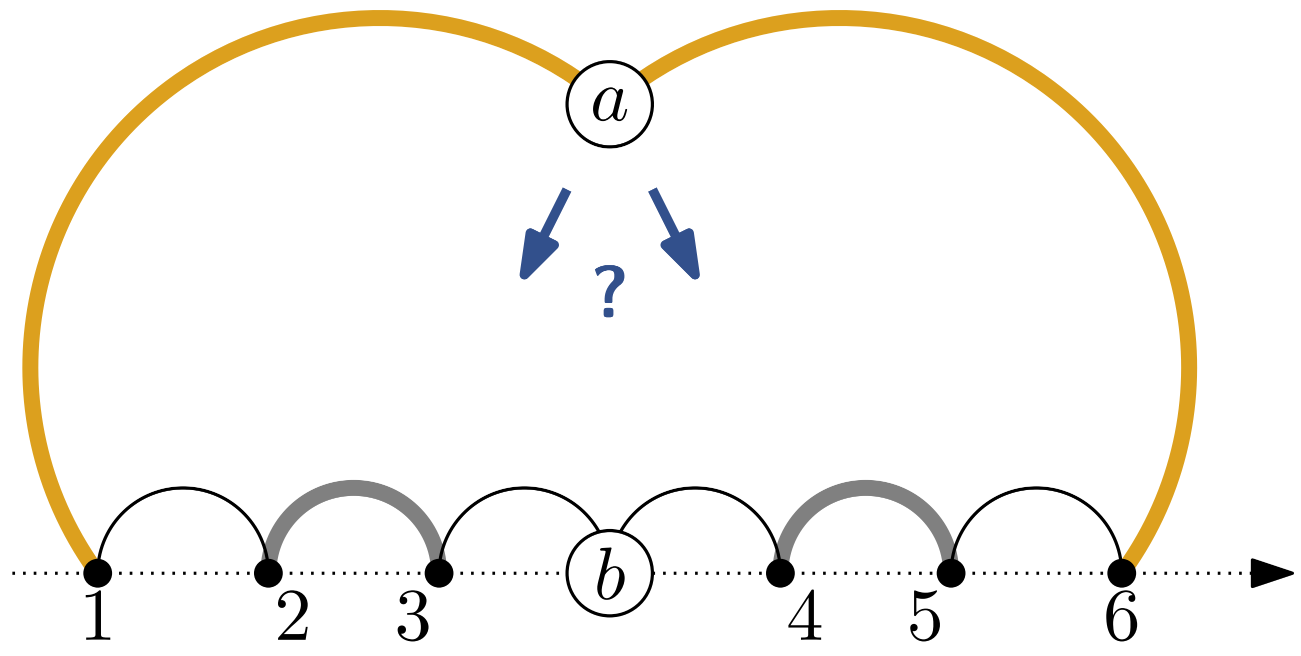
cover graph G_P

*also called $(k+1)$-rainbow

▷ **Queue-Number of a Poset** (Heath, Pemmaraju 1997).

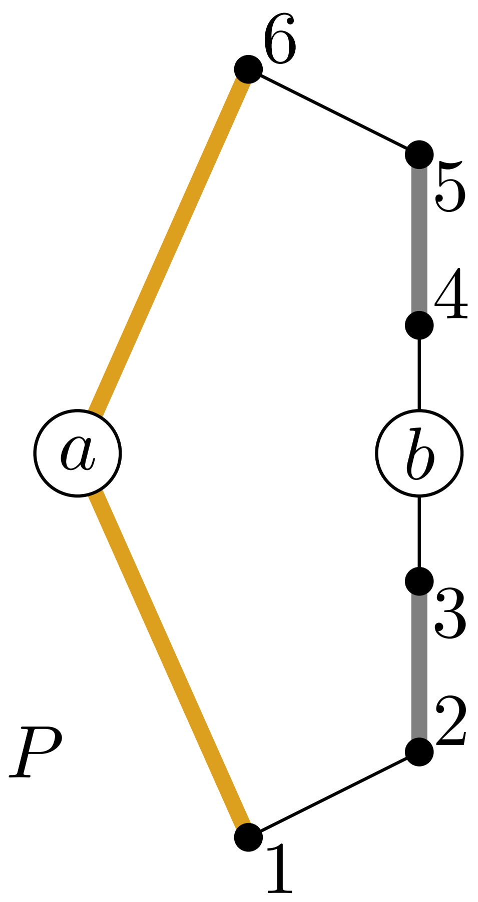
$\mathrm{qn}(P)$ = min k s.t. ∃ linear extension with no $(k+1)$-nesting of covers

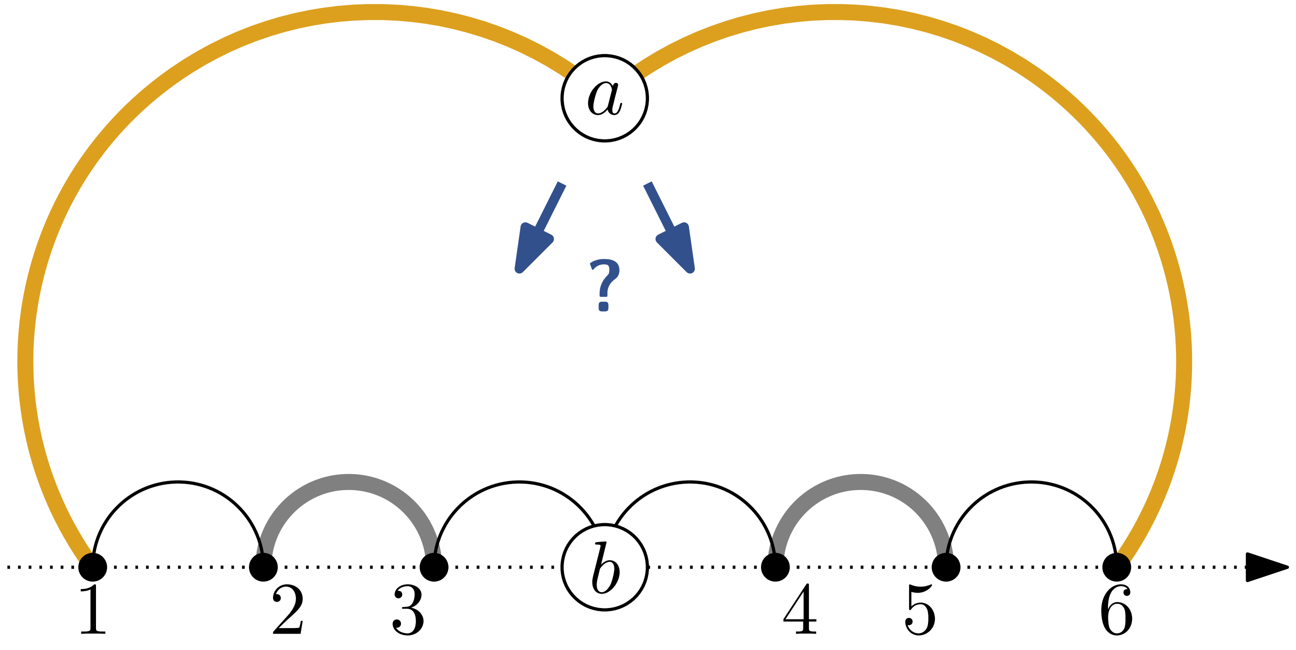




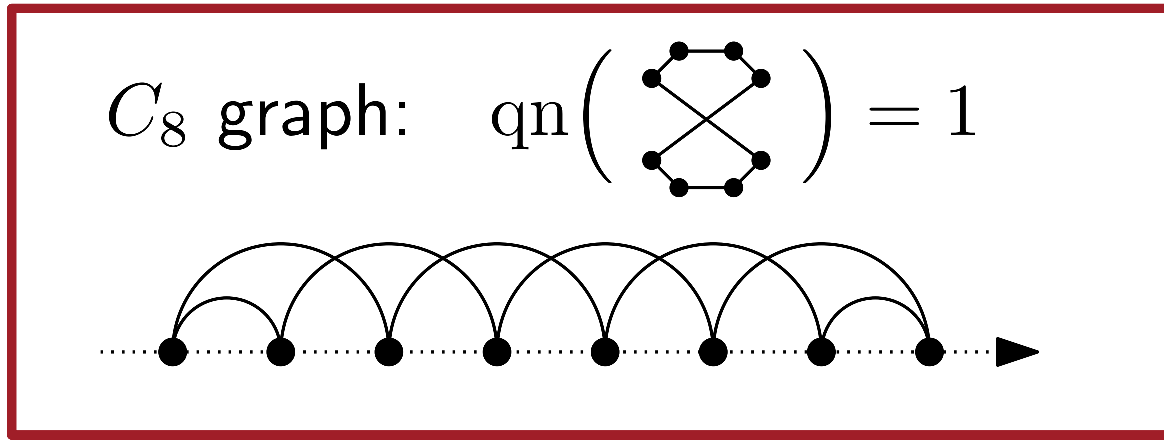
▷ **Queue-Number of a Poset** (Heath, Pemmaraju 1997).

$\mathrm{qn}(P)$ = min k s.t. ∃ linear extension with no $(k+1)$-nesting of covers





$\mathrm{qn}(P) = 2$



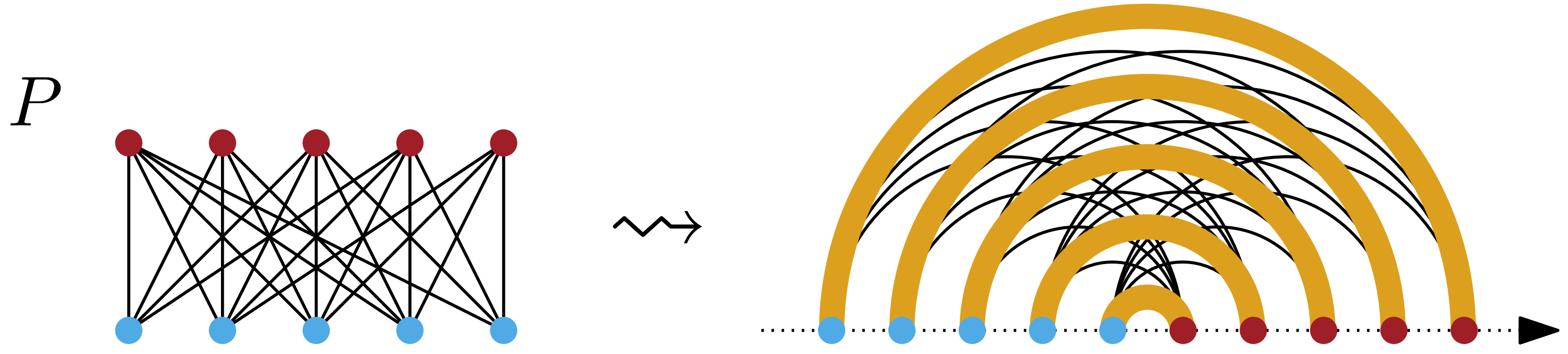
Theorem (Heath, Pemmaraju 1997).

P poset, $w = \mathrm{width}(P)$

▷ $\exists P$ with $\mathrm{qn}(P) \geq w$

▷ $\forall P$ $\mathrm{qn}(P) \leq w^2$

lower bound



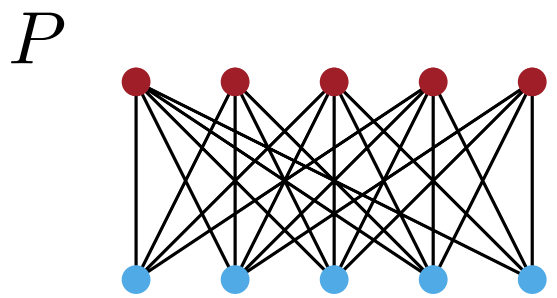
Theorem (Heath, Pemmaraju 1997).

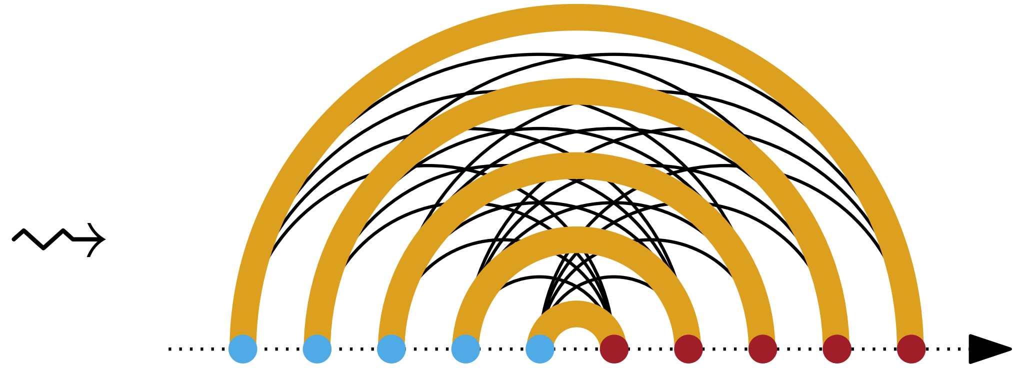
P poset, $w = \mathrm{width}(P)$

▷ $\exists P$ with $\mathrm{qn}(P) \geq w$

▷ $\forall P$ $\mathrm{qn}(P) \leq w^2$

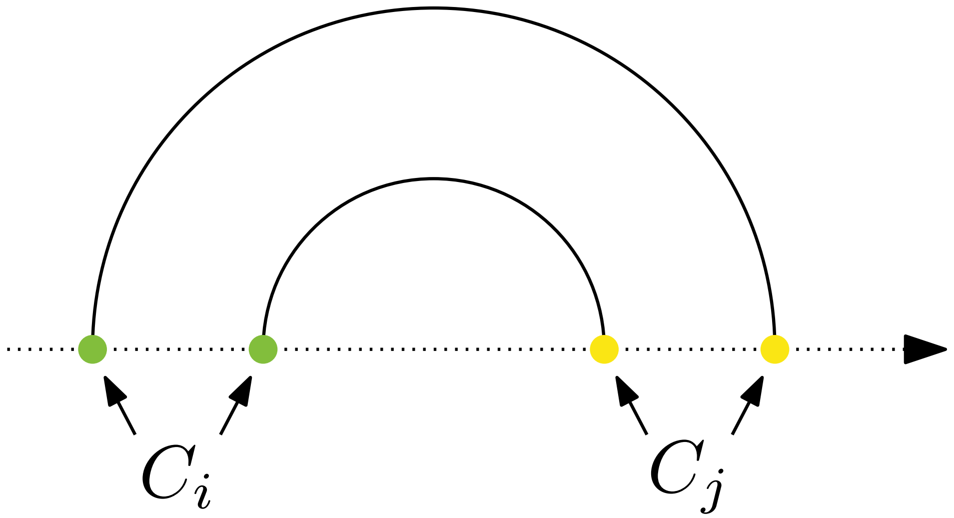
lower bound





upper bound

- $C_1, \ldots, C_w$ chain partition of P
- L *any* linear extension of P

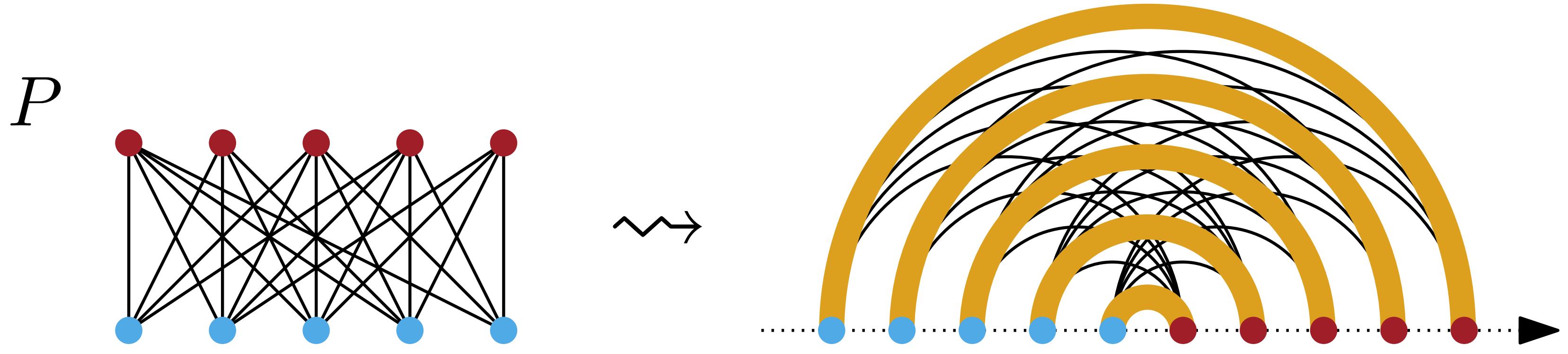


Theorem (Heath, Pemmaraju 1997).

P poset, $w = \mathrm{width}(P)$

- ▷ $\exists P$ with $\mathrm{qn}(P) \geq w$
- ▷ $\forall P$ $\mathrm{qn}(P) \leq w^2$

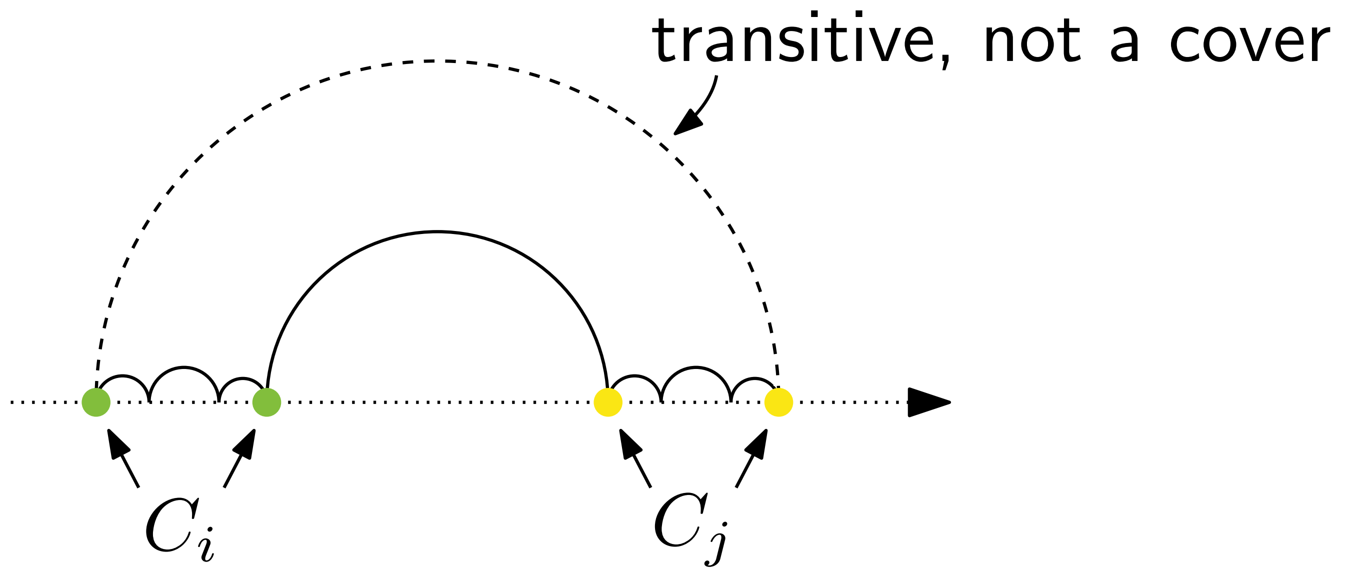
lower bound



upper bound

- $C_1, \ldots, C_w$ chain partition of P
- L *any* linear extension of P
- covers $u < v$ with $u \in C_i, v \in C_j$ form a queue $\forall i, j$

□



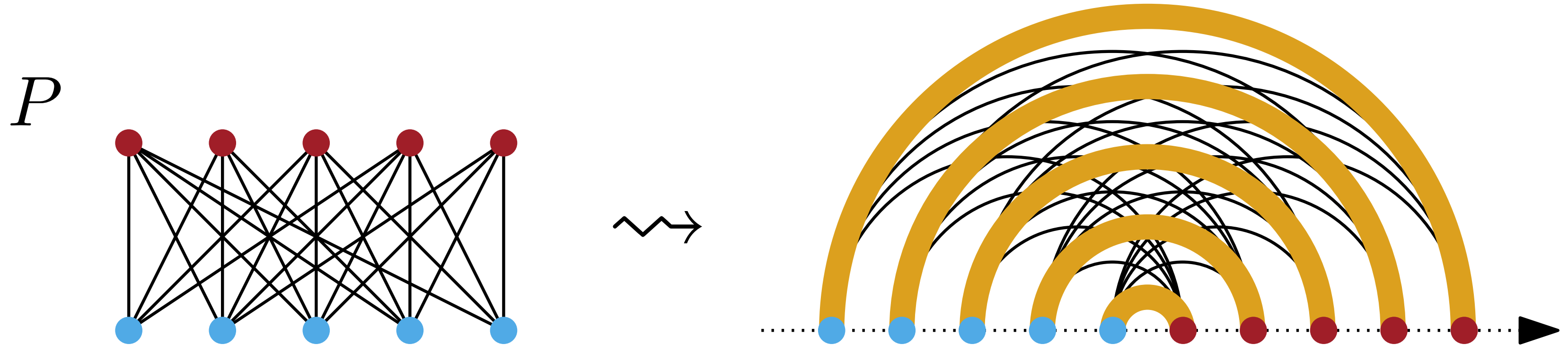
Theorem (Heath, Pemmaraju 1997).

P poset, $w = \mathrm{width}(P)$

▷ $\exists P$ with $\mathrm{qn}(P) \geq w$

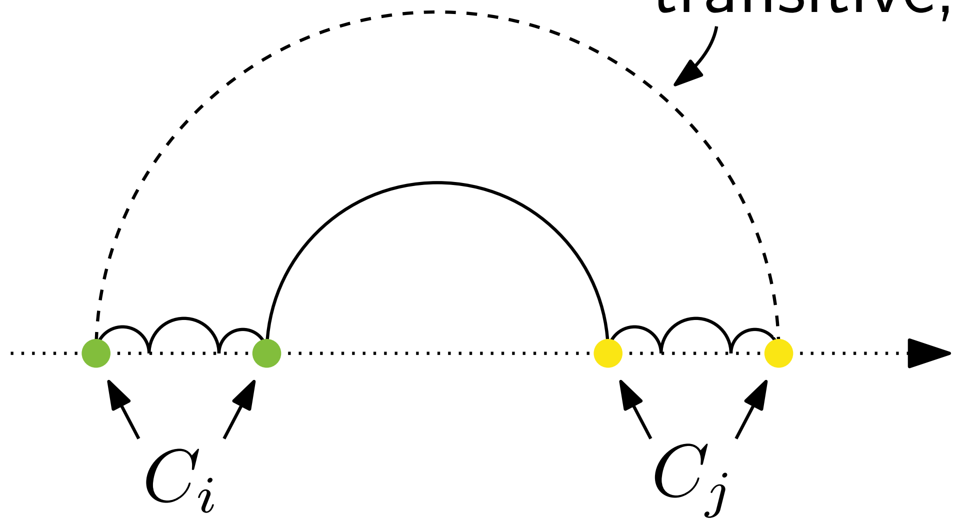
▷ $\forall P$ $\mathrm{qn}(P) \leq w^2$

lower bound



upper bound

- $C_1, \ldots, C_w$ chain partition of P
- L *any* linear extension of P
- covers $u < v$ with $u \in C_i, v \in C_j$ form a queue $\forall i, j$ □



Conjecture (Heath, Pemmaraju 1997).

▷ $\forall P$ $\mathrm{qn}(P) \leq w$

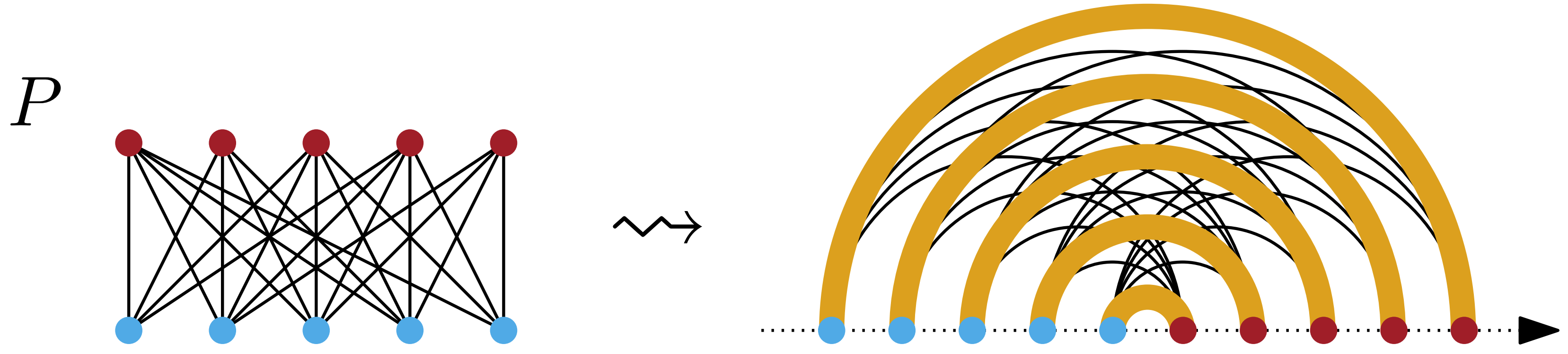
Theorem (Heath, Pemmaraju 1997).

P poset, $w = \mathrm{width}(P)$

▷ $\exists P$ with $\mathrm{qn}(P) \geq w$

▷ $\forall P$ $\mathrm{qn}(P) \leq w^2$

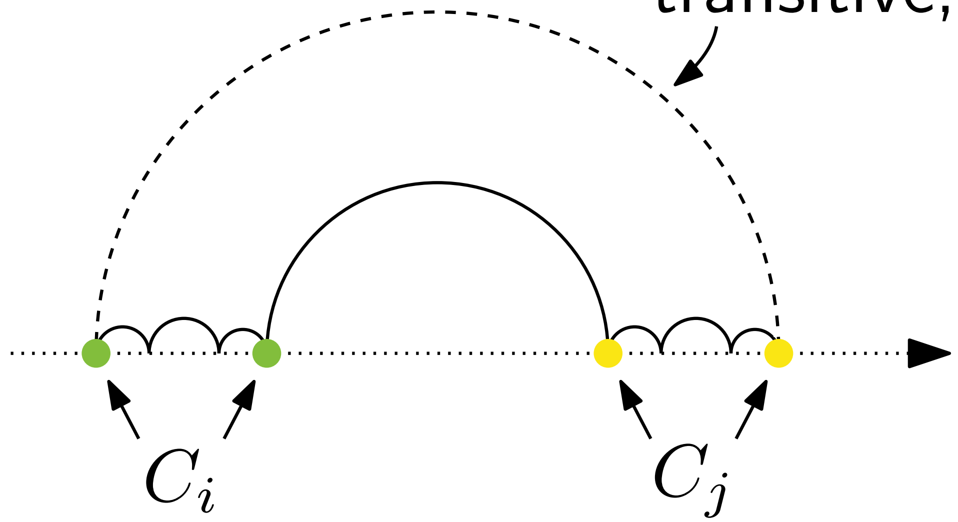
lower bound



upper bound

- $C_1, \ldots, C_w$ chain partition of P
- L *any* linear extension of P
- covers $u < v$ with $u \in C_i, v \in C_j$ form a queue $\forall i, j$

□



Conjecture (Heath, Pemmaraju 1997).

▷ $\forall P$ $\mathrm{qn}(P) \leq w$

FALSE

Theorem (Alam et al., GD 2020).

▷ $\forall w \geq 3\ \exists P_w$ $\mathrm{qn}(P_w) \geq w + 1$

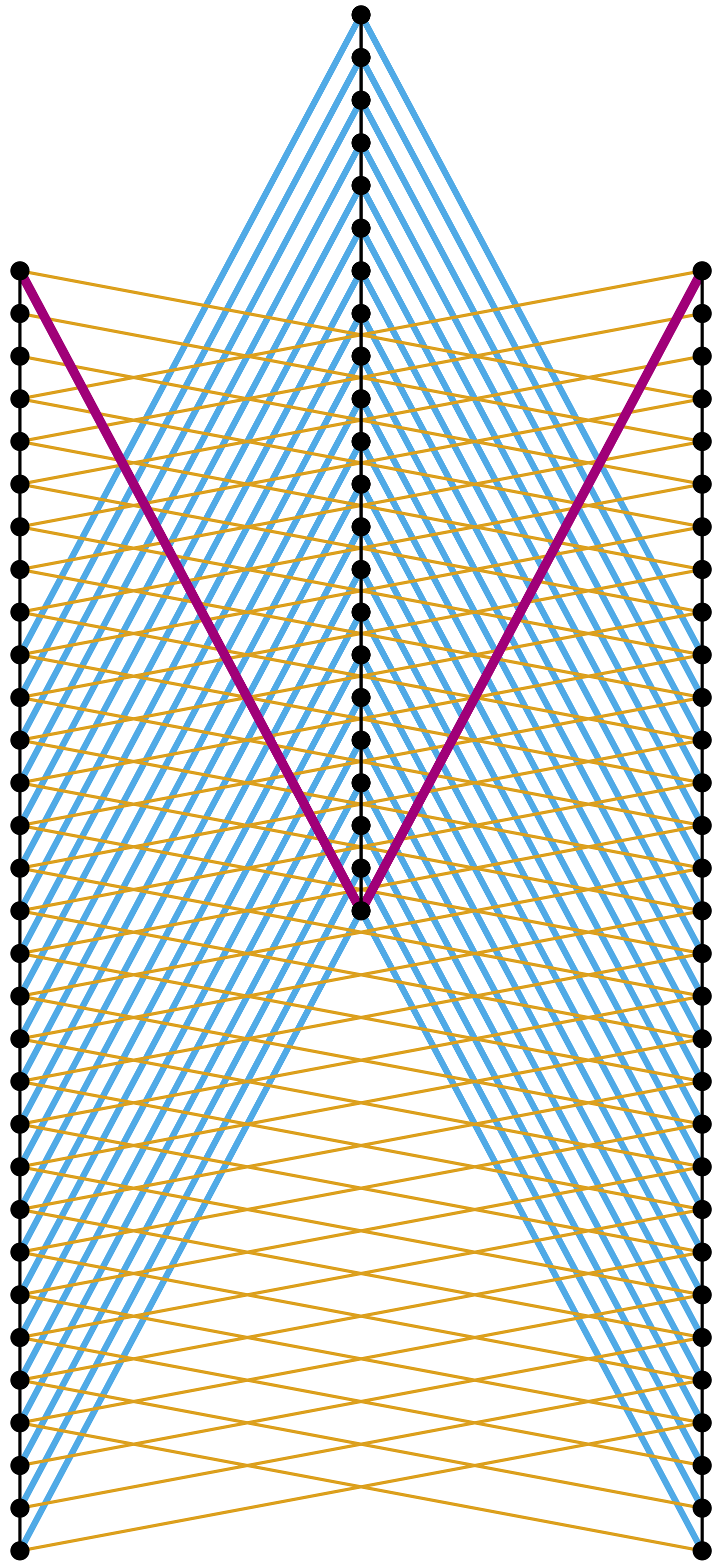
Theorem (Alam et al., GD 2020).

▷ $\forall w \geq 3 \; \exists P_w \qquad \mathrm{qn}(P_w) \geq w + 1$

step 1

construct poset P_3 s.t.

- $\mathrm{width}(P_3) = 3$
- $\mathrm{qn}(P_3) = 4$



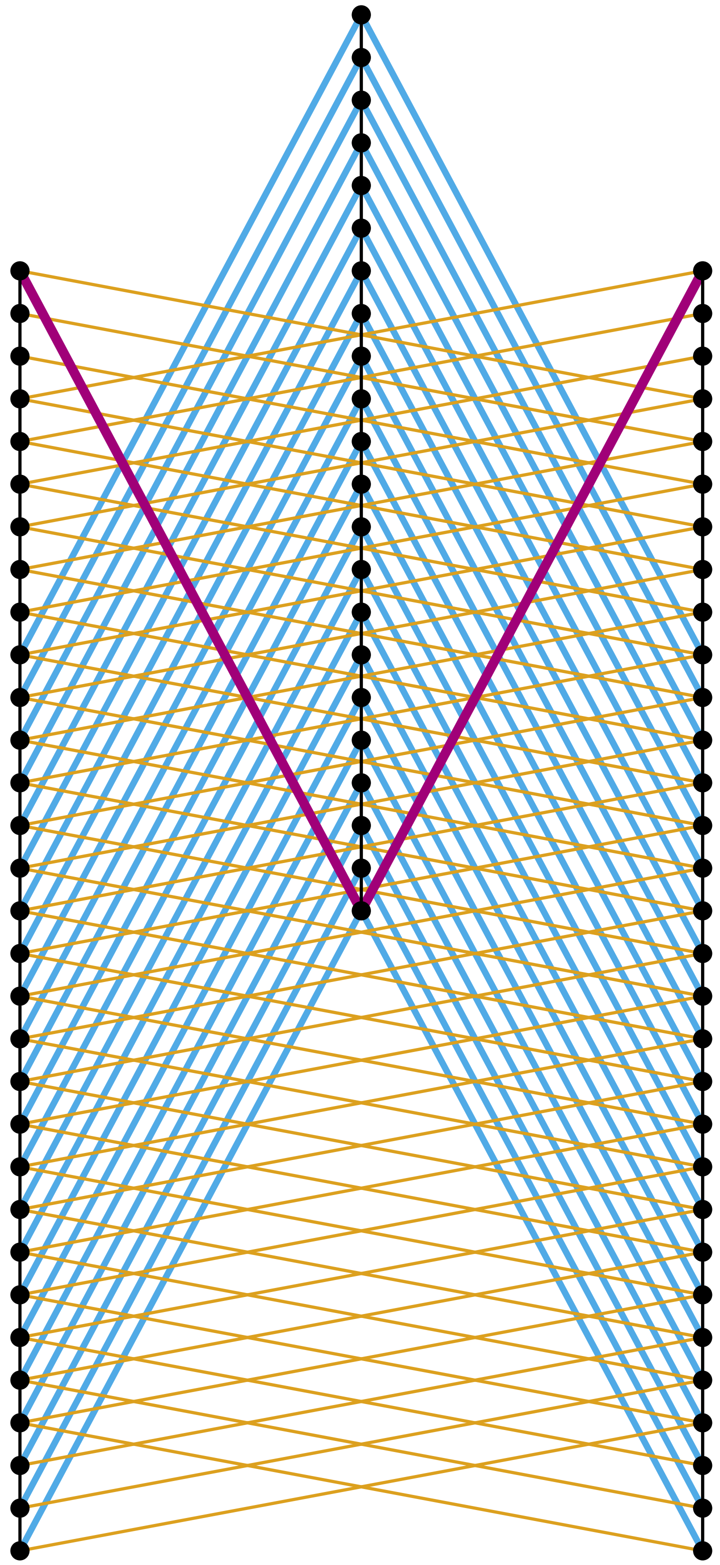
Theorem (Alam et al., GD 2020).

▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w + 1$

step 1

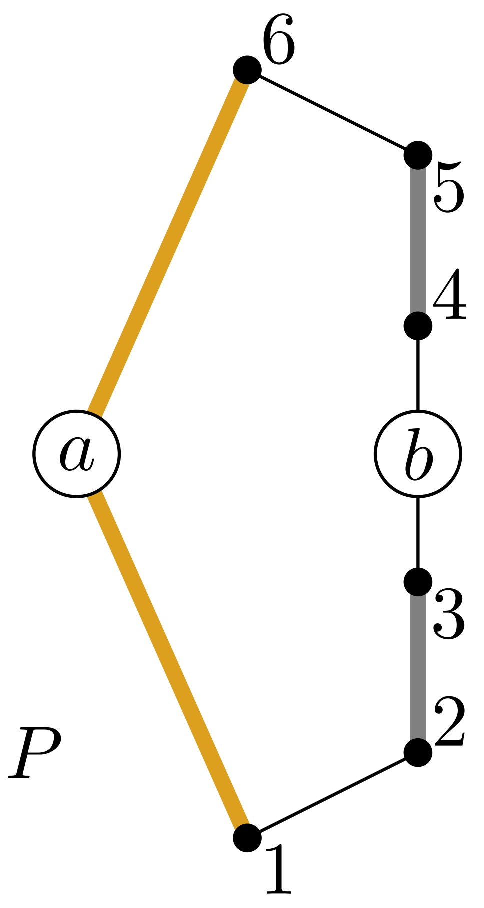
construct poset P_3 s.t.

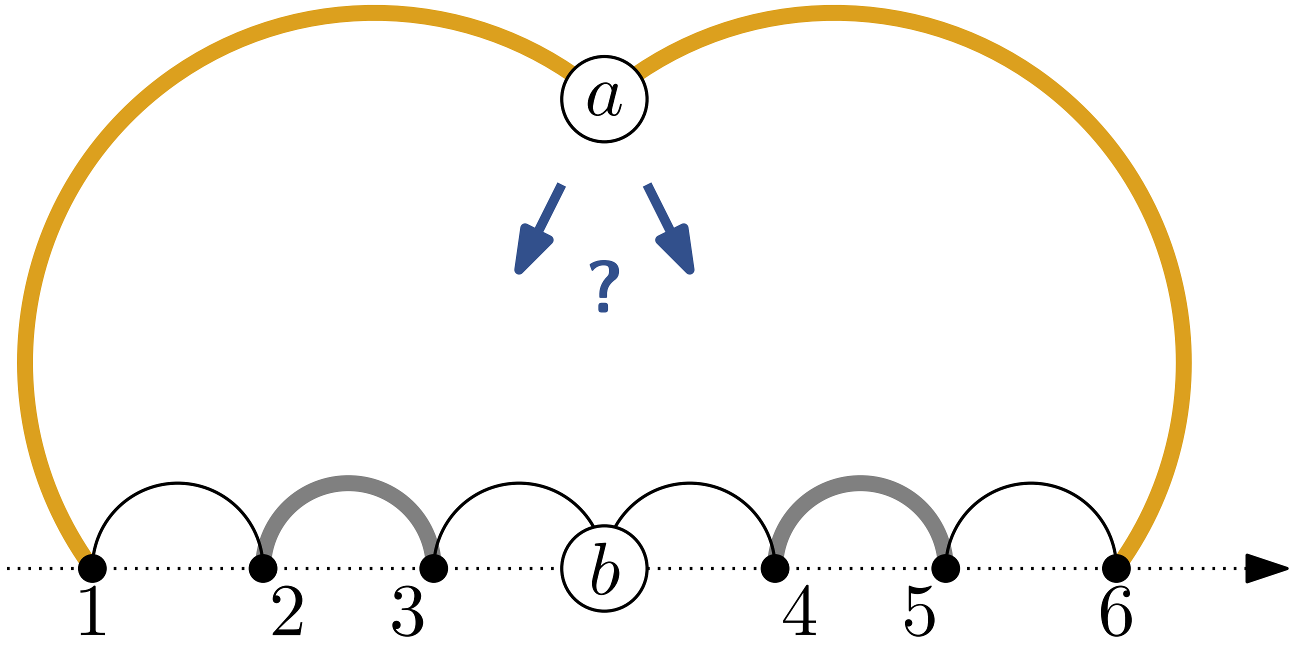
- $\mathrm{width}(P_3) = 3$
- $\mathrm{qn}(P_3) = 4$



step 2

construct P_w from P_{w-1}





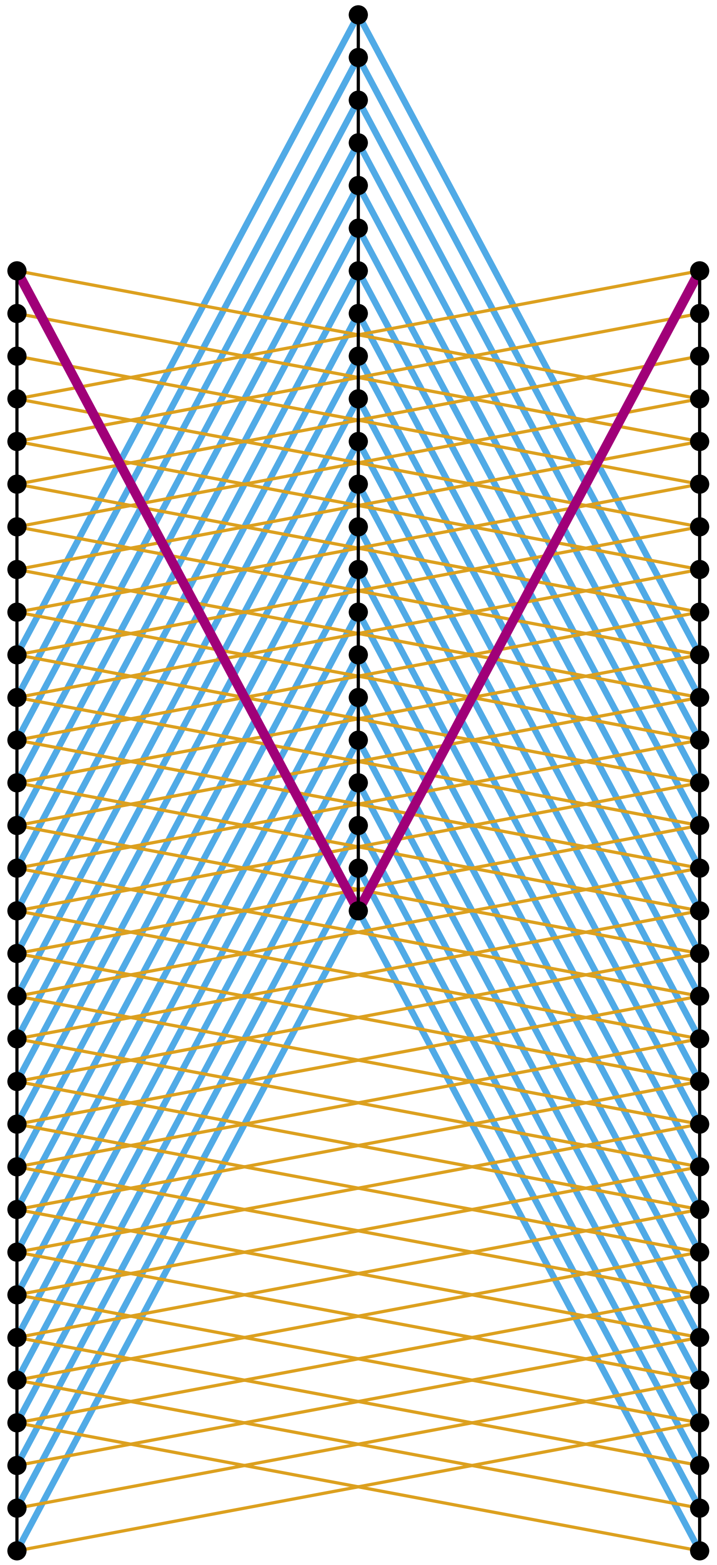
Theorem (Alam et al., GD 2020).

▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w+1$

<u>step 1</u>

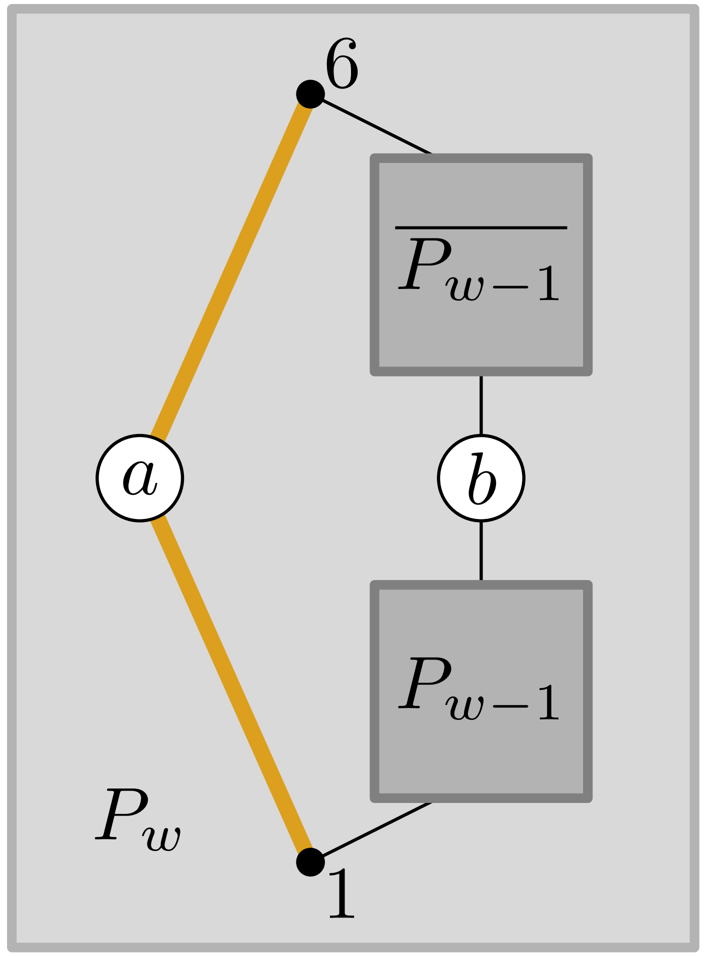
construct poset P_3 s.t.

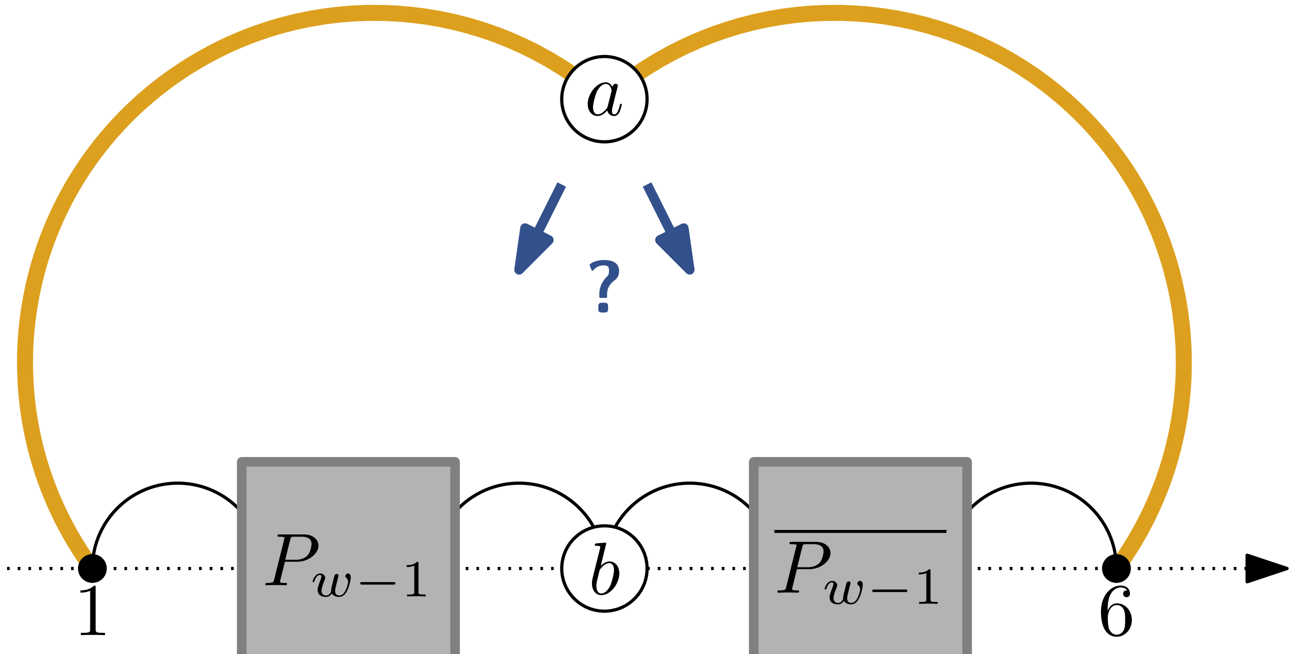
- $\mathrm{width}(P_3) = 3$
- $\mathrm{qn}(P_3) = 4$



<u>step 2</u>

construct P_w from P_{w-1}





Heath, Pemmaraju (1997)

$\forall w \; \forall P_w \qquad \text{qn}(P_w) \leq w^2$

Knauer, Micek, U. (GD 2018)

$\forall P_2 \qquad \text{qn}(P_2) \leq 2$

Alam et al. (GD 2020)

$\forall w \geq 3 \; \exists P_w \qquad \text{qn}(P_w) \geq w + 1$

Heath, Pemmaraju (1997)

$\forall w \; \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

Knauer, Micek, U. (GD 2018)

$\forall P_2 \qquad \mathrm{qn}(P_2) \leq 2$

Alam et al. (GD 2020)

$\forall w \geq 3 \; \exists P_w \qquad \mathrm{qn}(P_w) \geq w + 1$

Theorem.

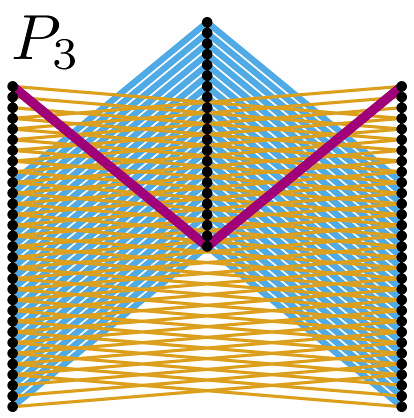
▷ $\forall w \geq 3 \; \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

Heath, Pemmaraju (1997)
$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

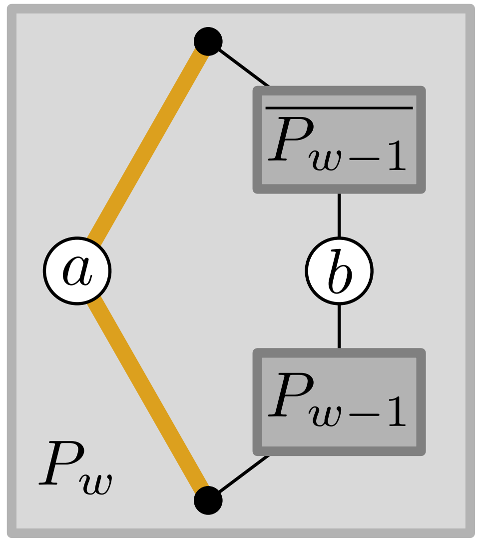
Knauer, Micek, U. (GD 2018)
$\forall P_2 \qquad \mathrm{qn}(P_2) \leq 2$

Alam et al. (GD 2020)
$\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w+1$

step 1

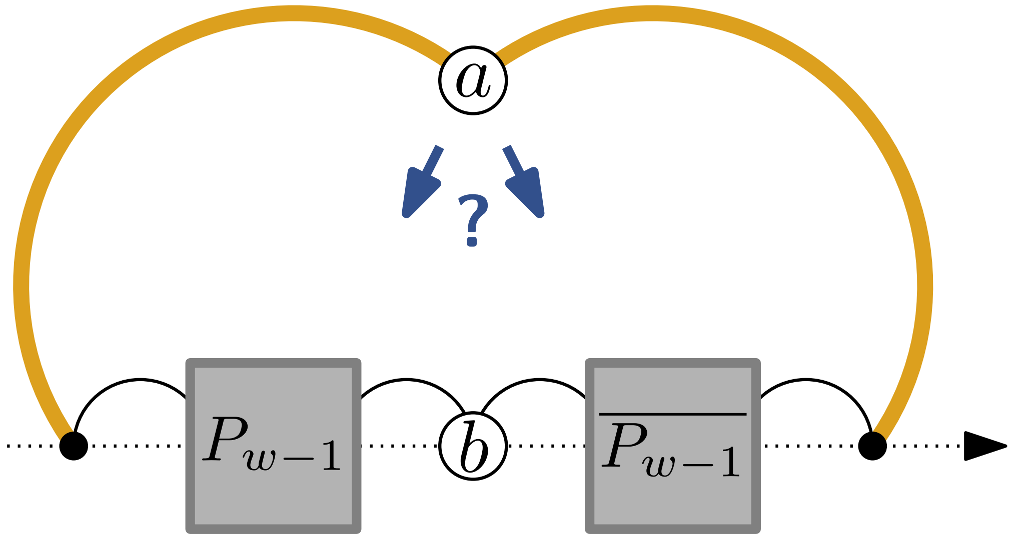


step 2



Theorem.

▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

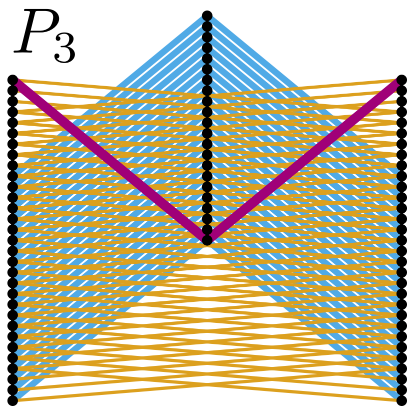


Heath, Pemmaraju (1997)
$\forall w \; \forall P_w$ $\qquad \mathrm{qn}(P_w) \leq w^2$

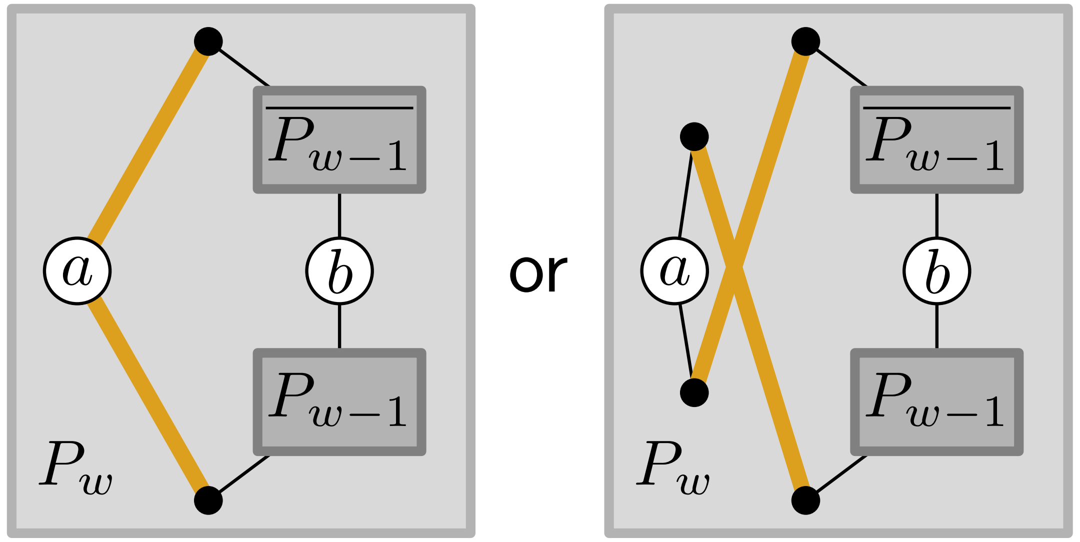
Knauer, Micek, U. (GD 2018)
$\forall P_2$ $\qquad \mathrm{qn}(P_2) \leq 2$

Alam et al. (GD 2020)
$\forall w \geq 3 \; \exists P_w$ $\qquad \mathrm{qn}(P_w) \geq w + 1$

step 1

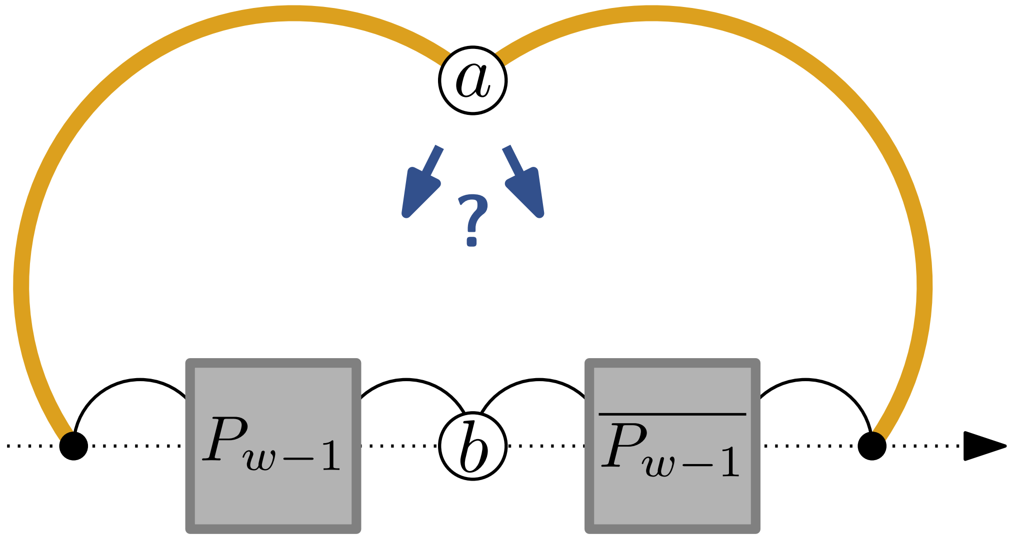


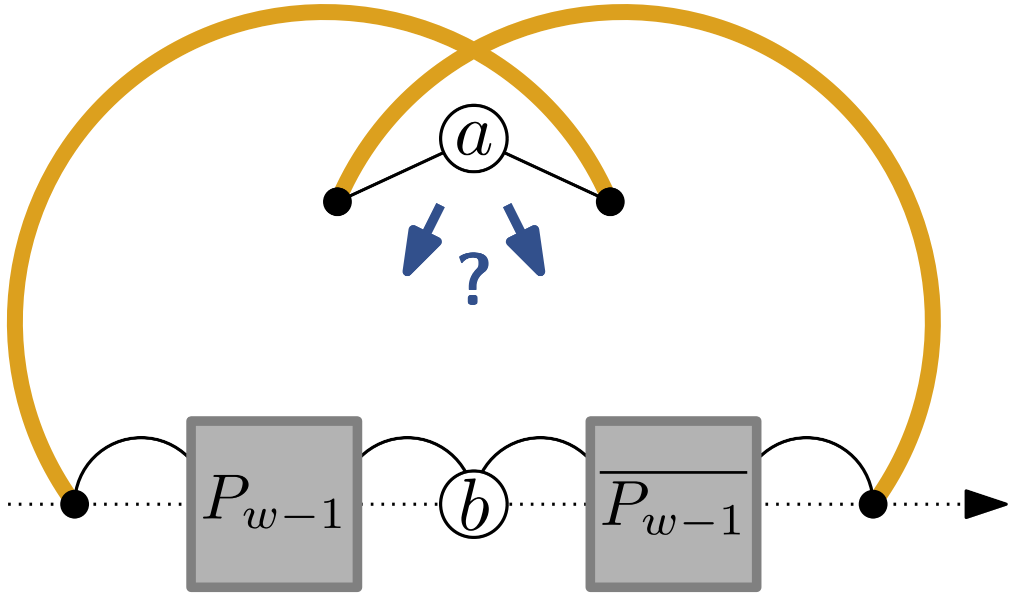
step 2



Theorem.

▷ $\forall w \geq 3 \; \exists P_w$ $\qquad \mathrm{qn}(P_w) \geq w^2/8$



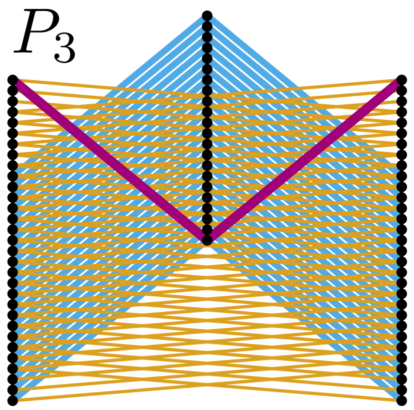


Heath, Pemmaraju (1997)
$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

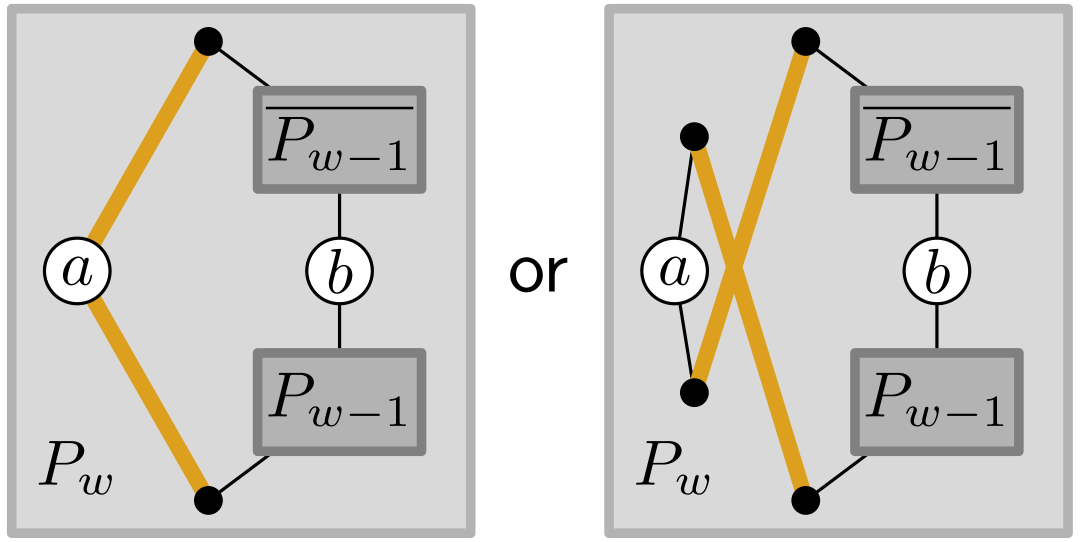
Knauer, Micek, U. (GD 2018)
$\forall P_2 \qquad \mathrm{qn}(P_2) \leq 2$

Alam et al. (GD 2020)
$\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w+1$

step 1

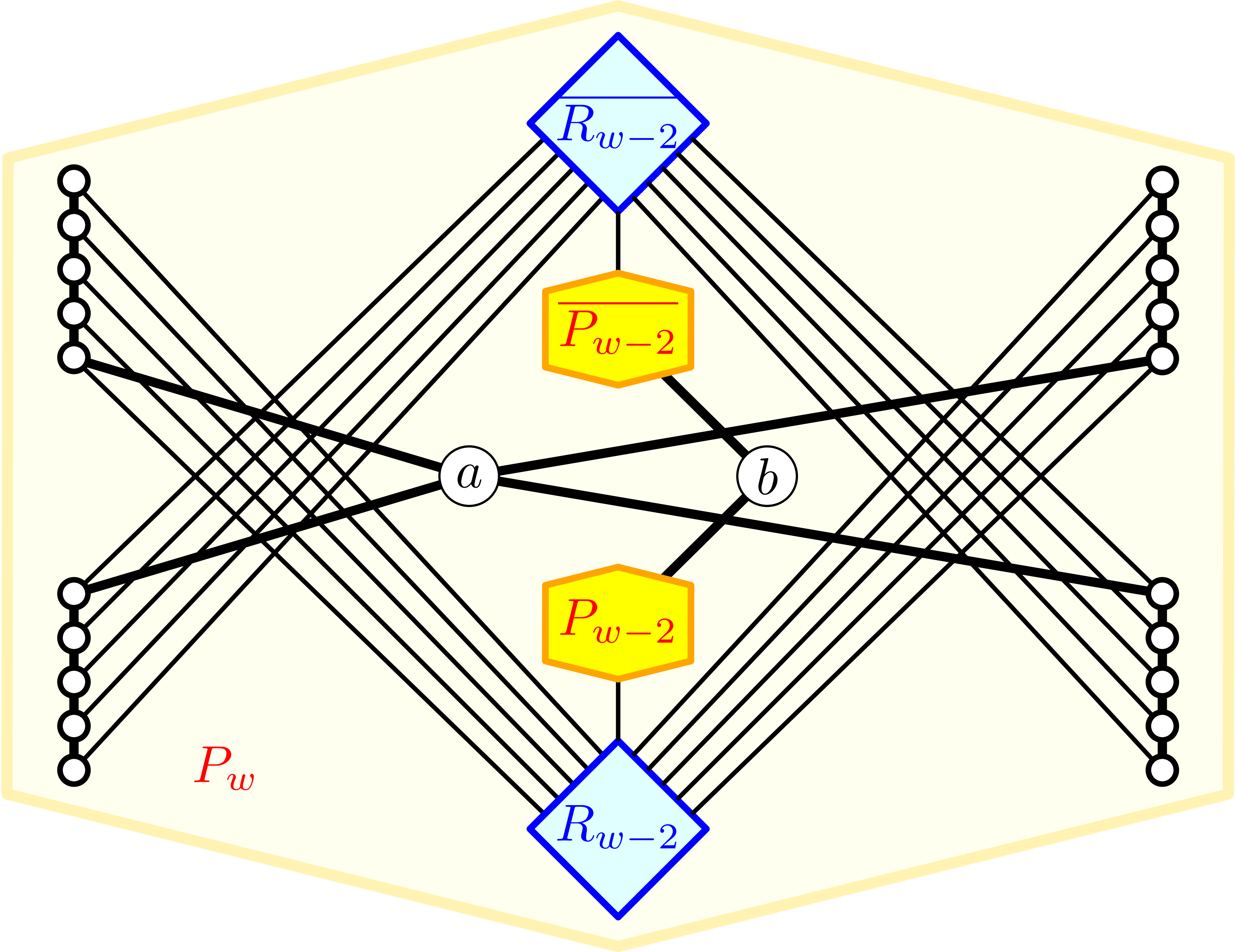


step 2



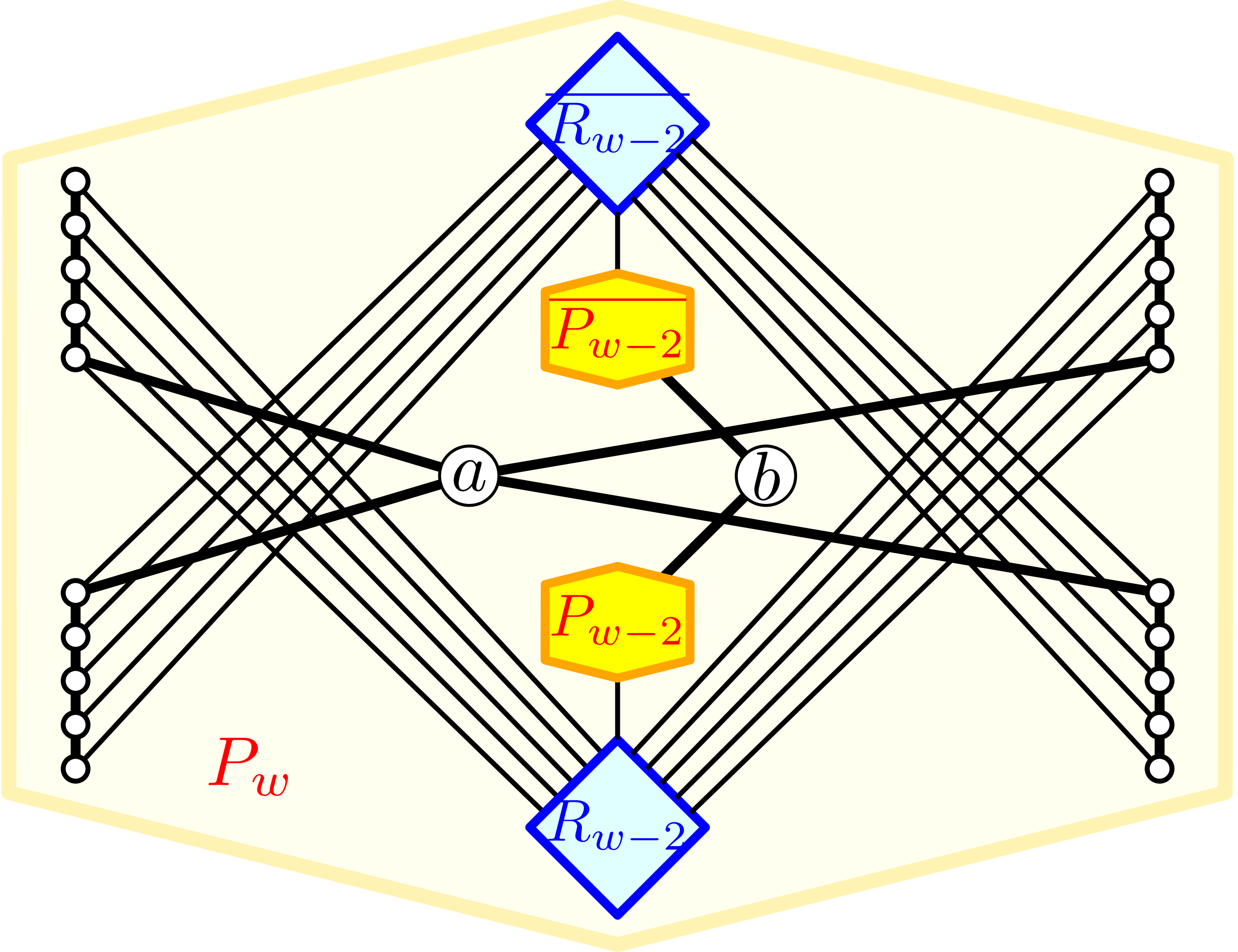
Theorem.

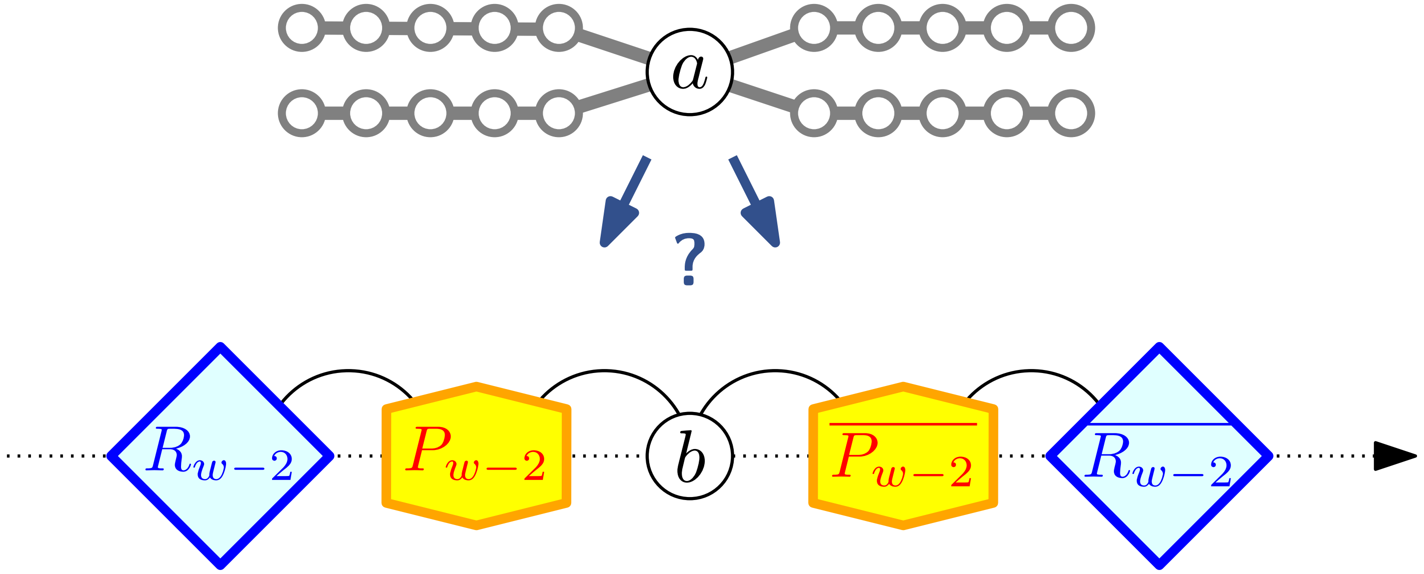
▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$



Theorem.

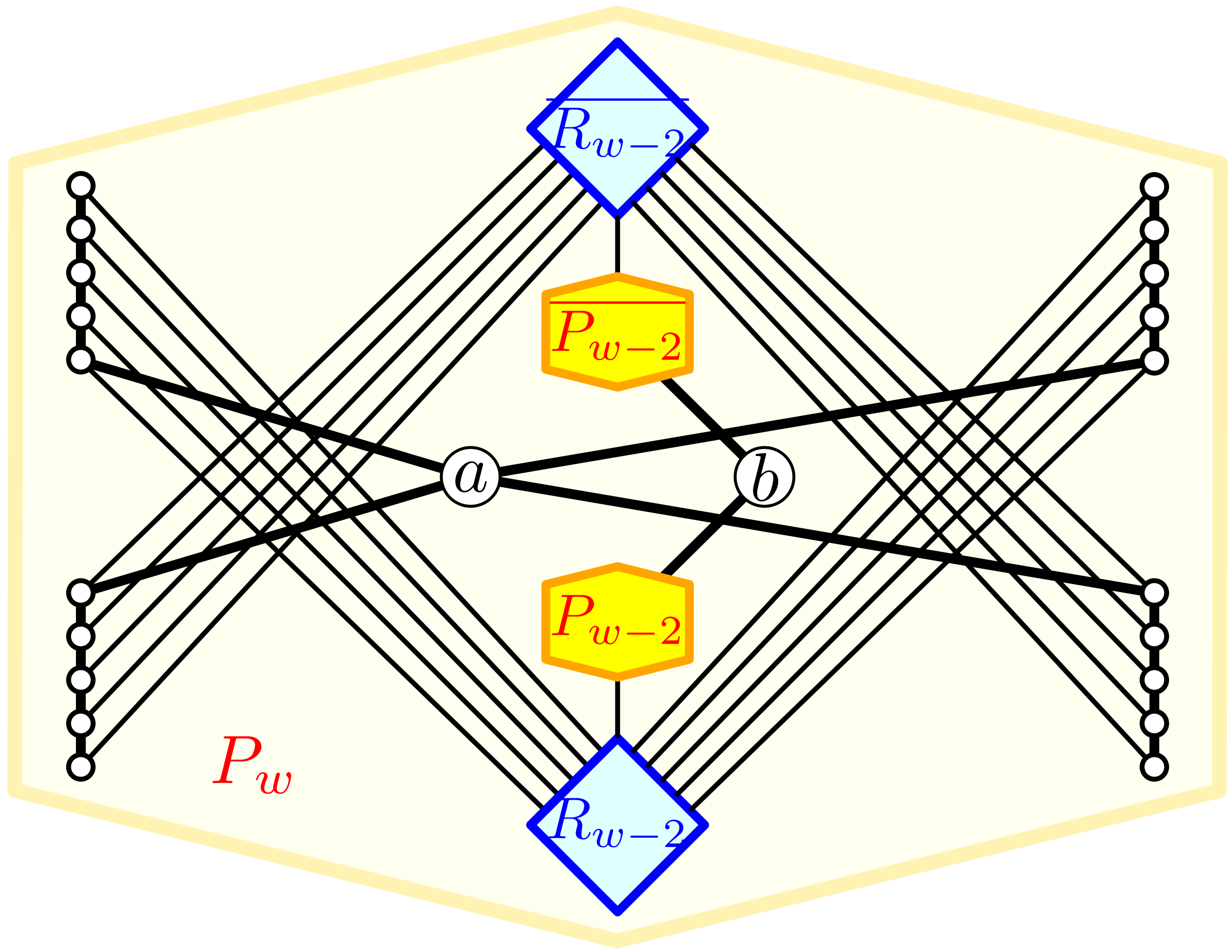
▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

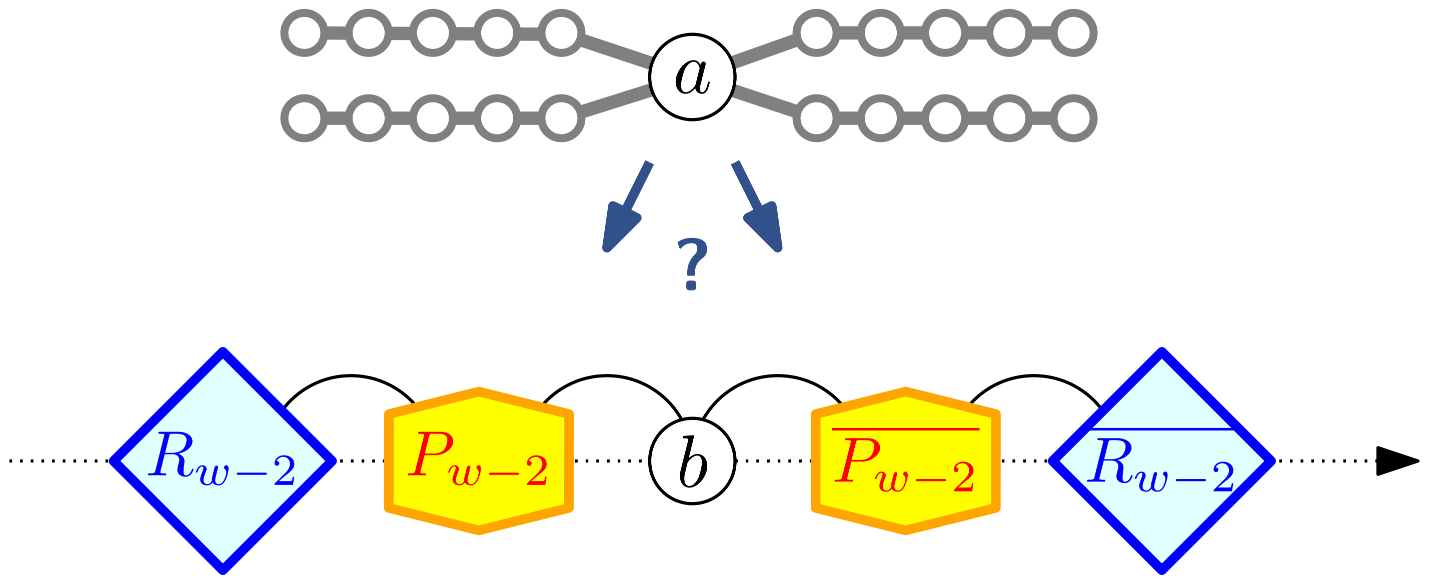




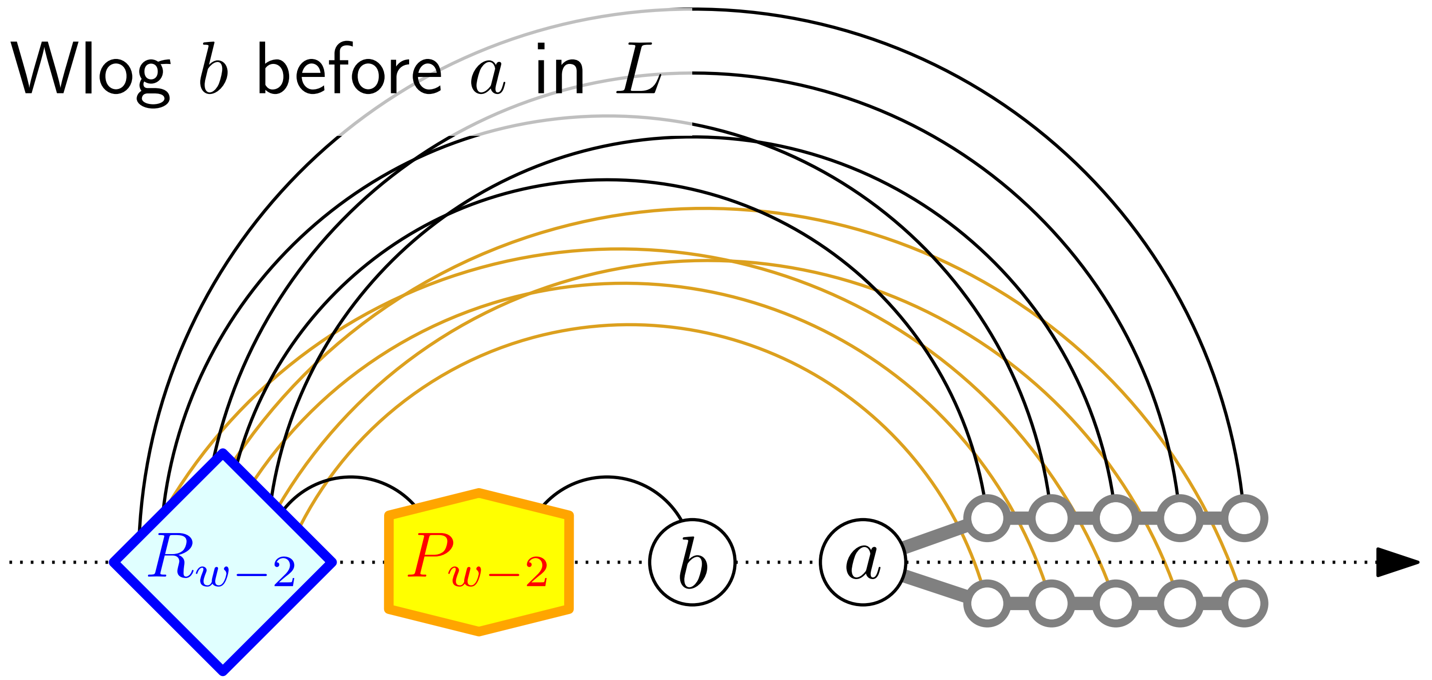
Theorem.

▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$



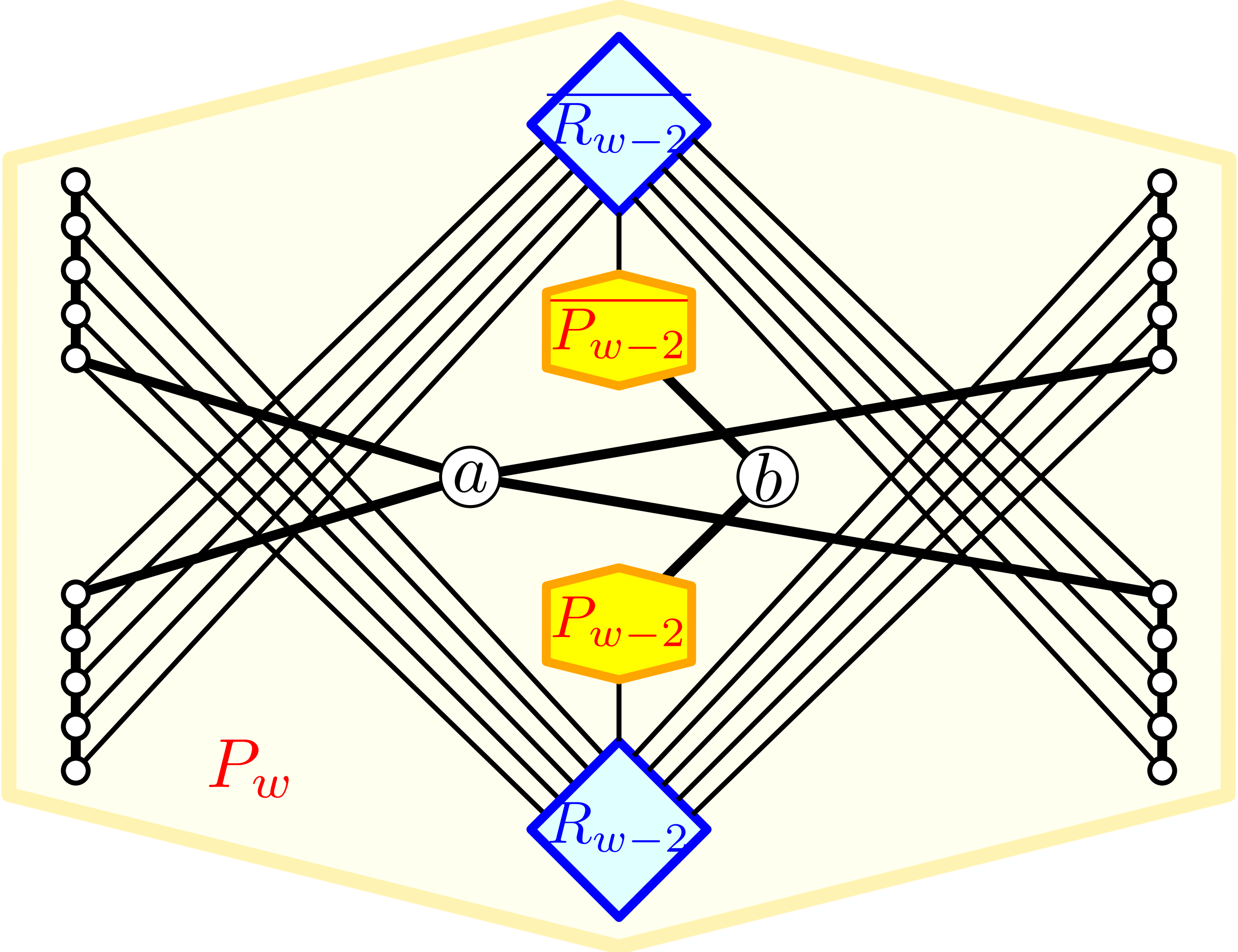


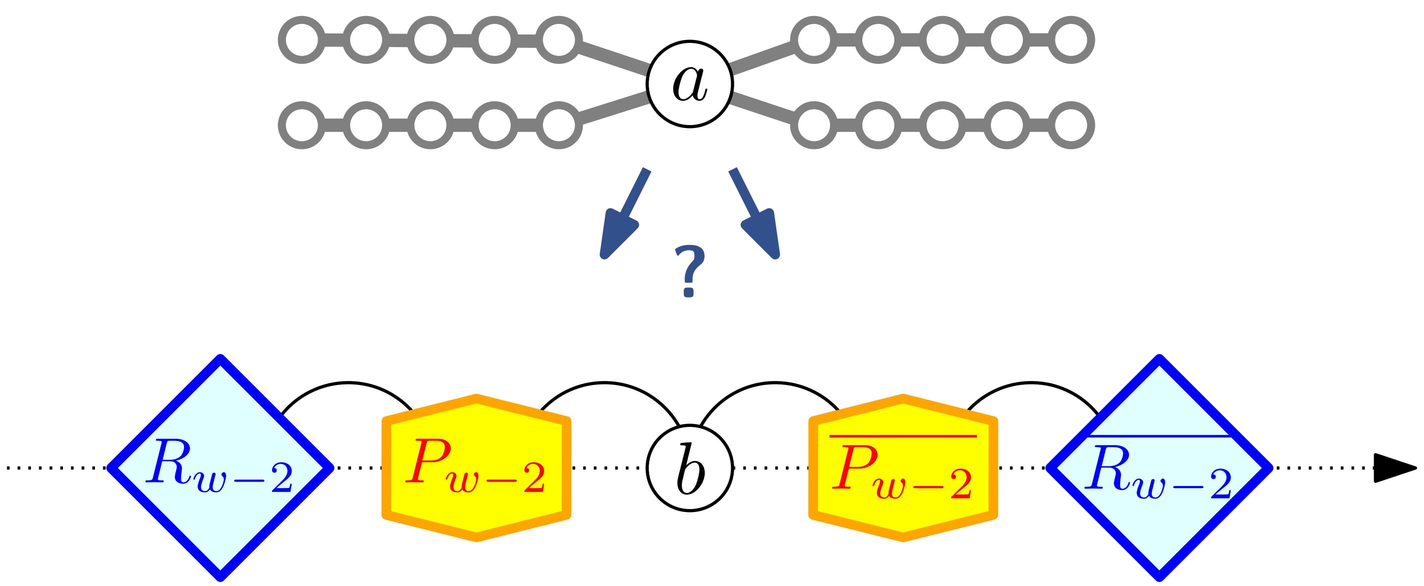
Wlog b before a in L



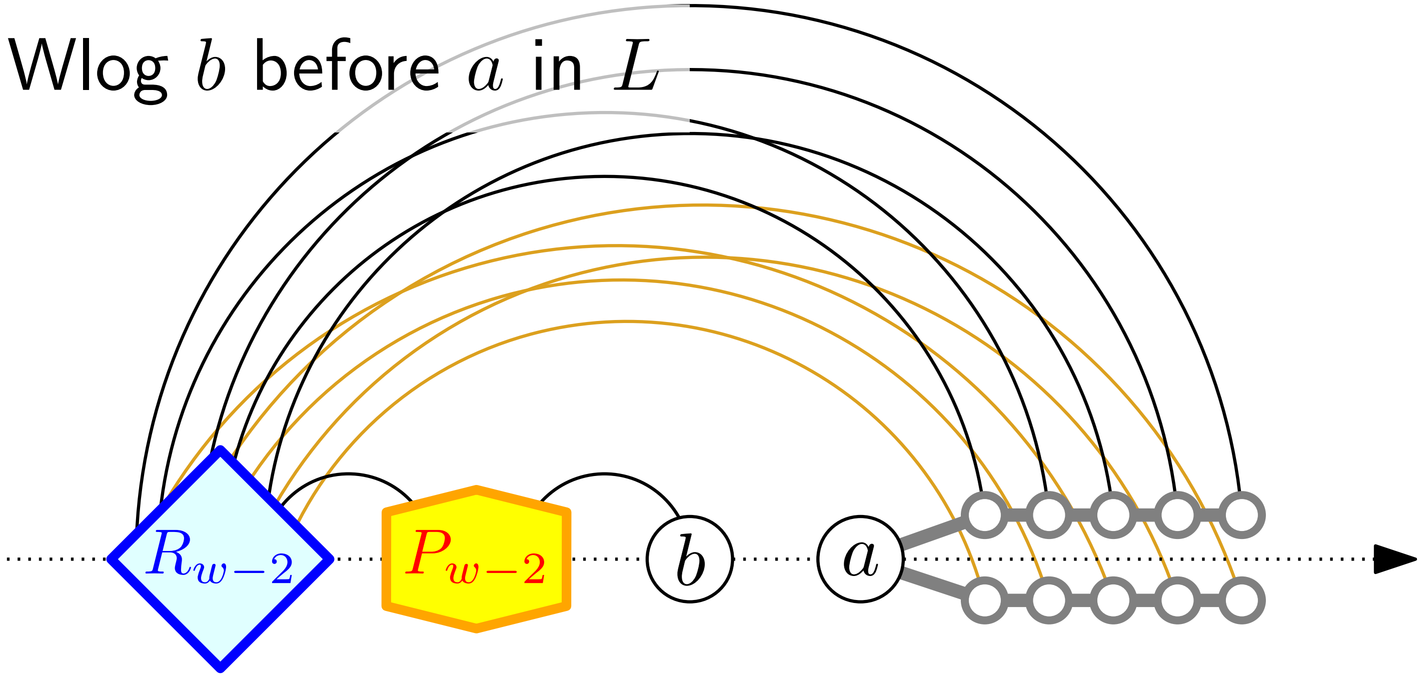
Theorem.

▷ $\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$





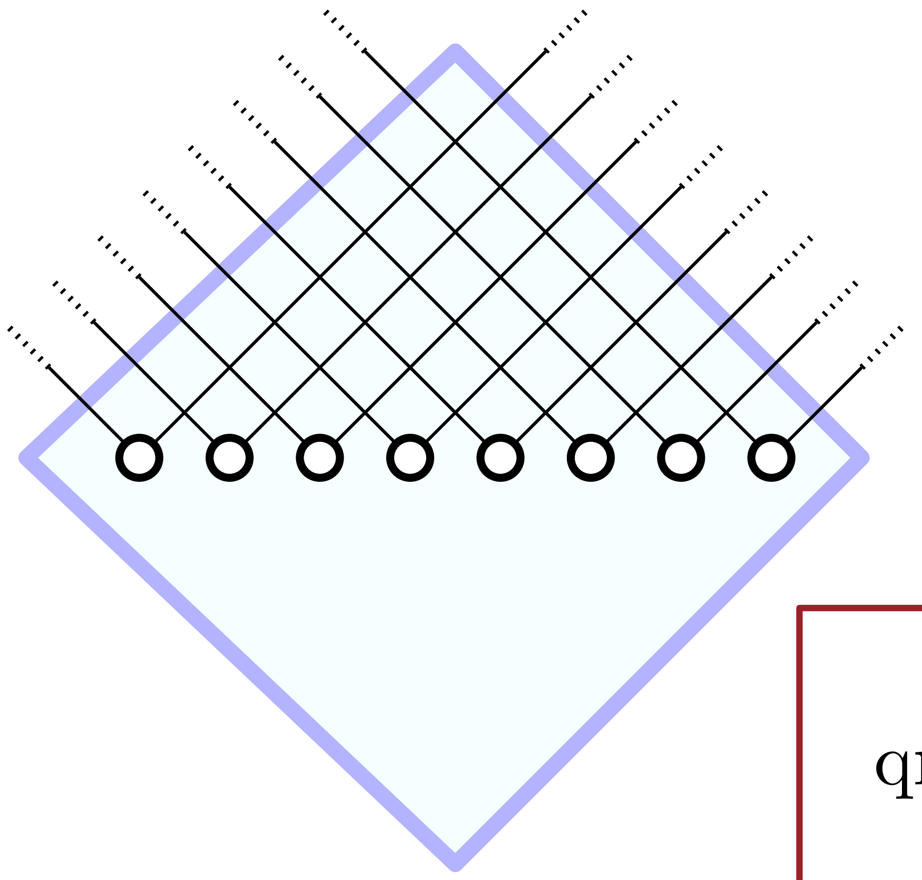
Wlog b before a in L

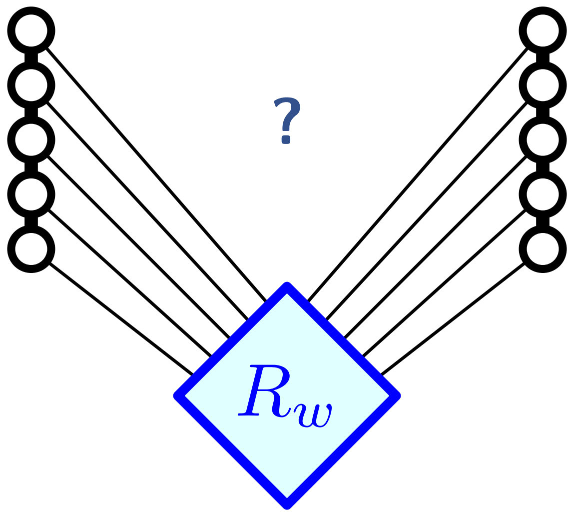


$$\mathrm{qn}(P_w) \geq \mathrm{qn}(P_{w-2}) + r_{w-2} \geq \sum_{u=1}^{w/2} r_{2u}$$

$\Longleftarrow$ r_{w-2} = size of largest forced such rainbow

first attempt for R_w



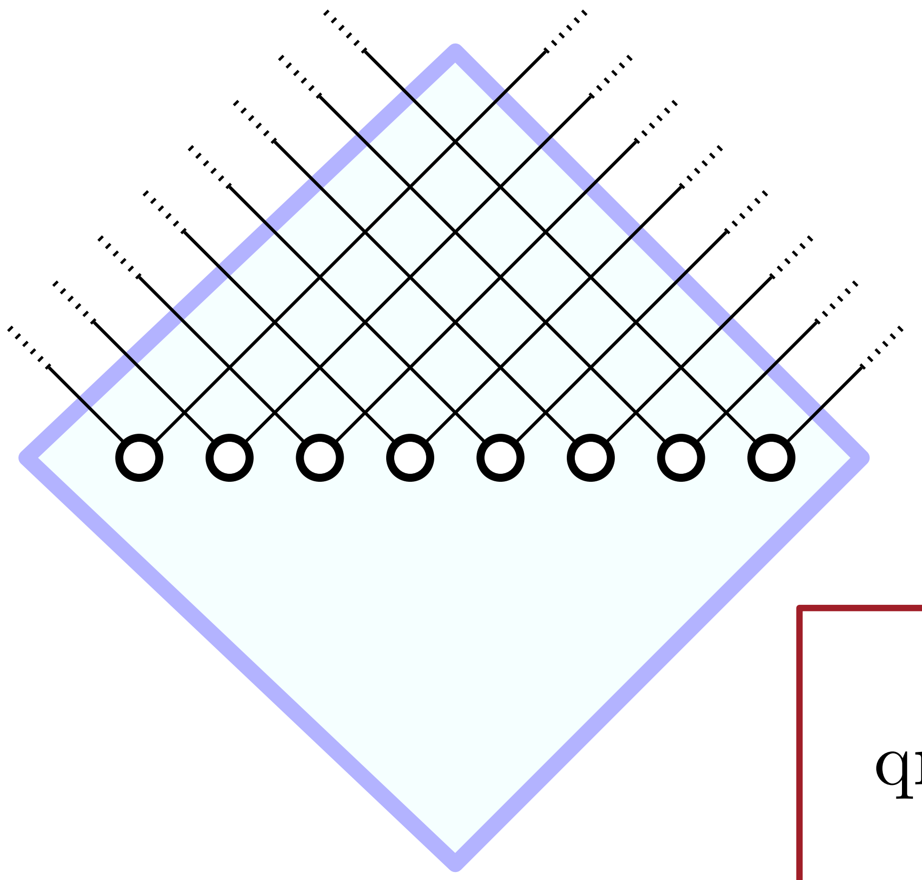


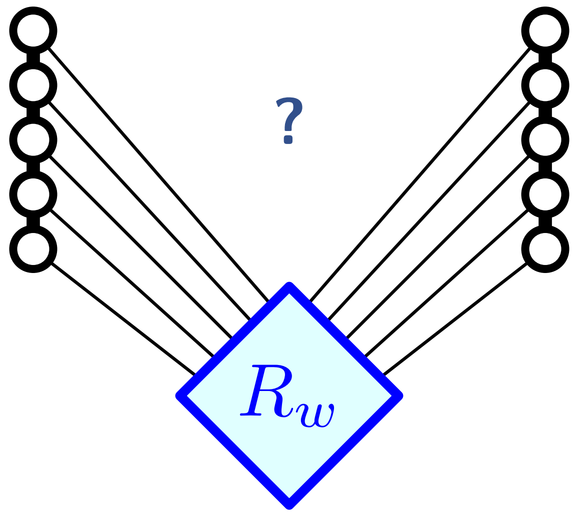
$$\mathrm{qn}(P_w) \geq \mathrm{qn}(P_{w-2}) + r_{w-2} \geq \sum_{u=1}^{w/2} r_{2u}$$

- Erdős-Szekeres gives $r_w \geq \sqrt{w}$

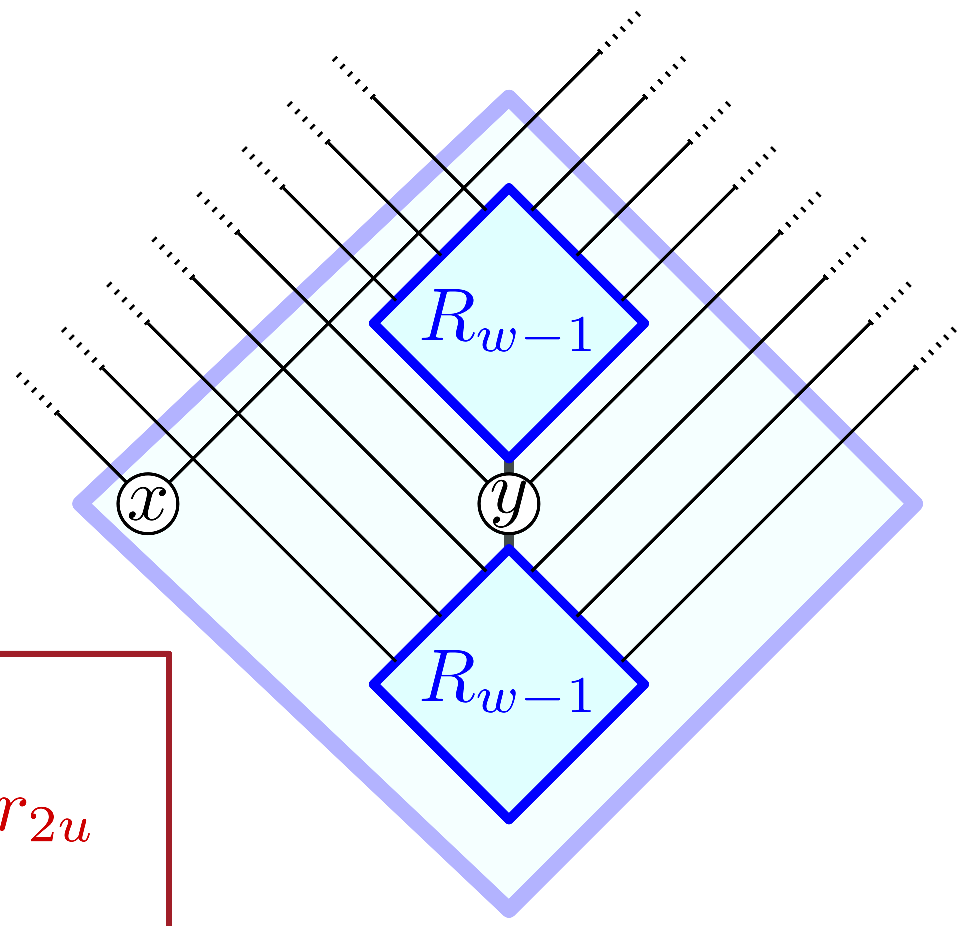
$$\Rightarrow \mathrm{qn}(P_w) \geq \sum_{u=1}^{w/2} \sqrt{2u} = \Omega(w^{3/2})$$

first attempt for R_w





the actual construction of R_w



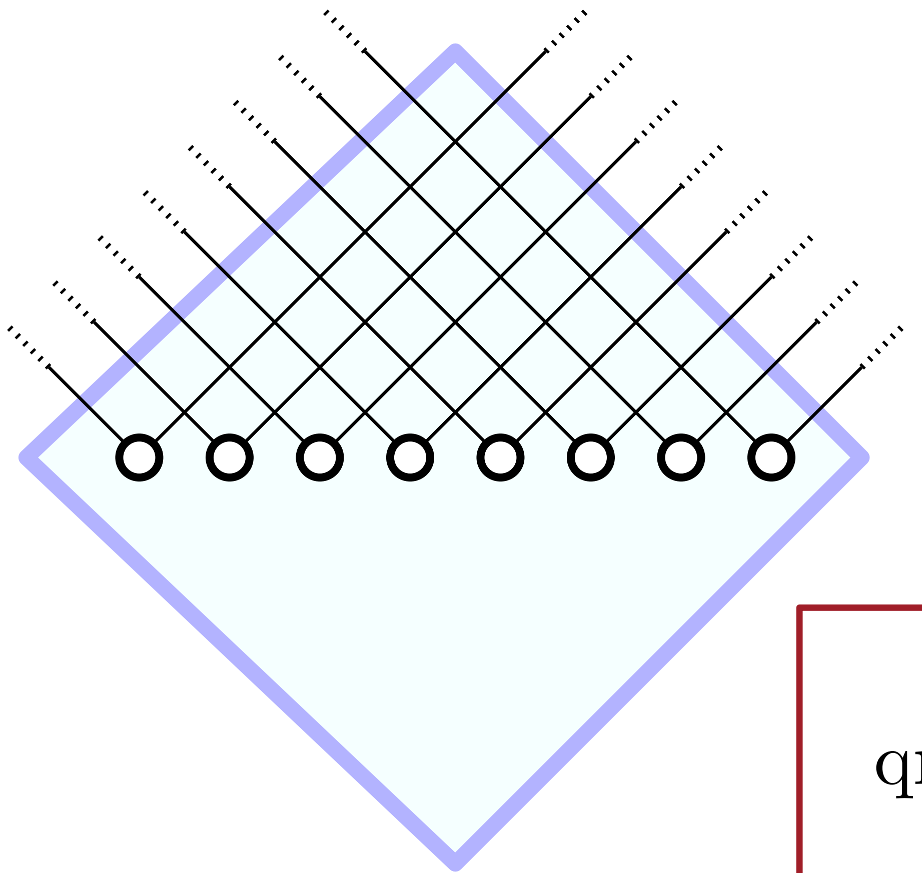
$$\mathrm{qn}(P_w) \geq \mathrm{qn}(P_{w-2}) + r_{w-2} \geq \sum_{u=1}^{w/2} r_{2u}$$

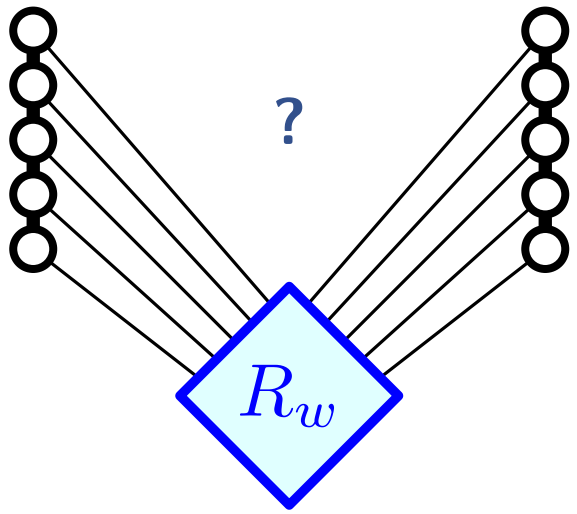
Claim: $\forall L\colon \mathrm{width}(L \cap L_1) + \mathrm{width}(L \cap L_2) \geq w + 1$

- Erdős-Szekeres gives $r_w \geq \sqrt{w}$

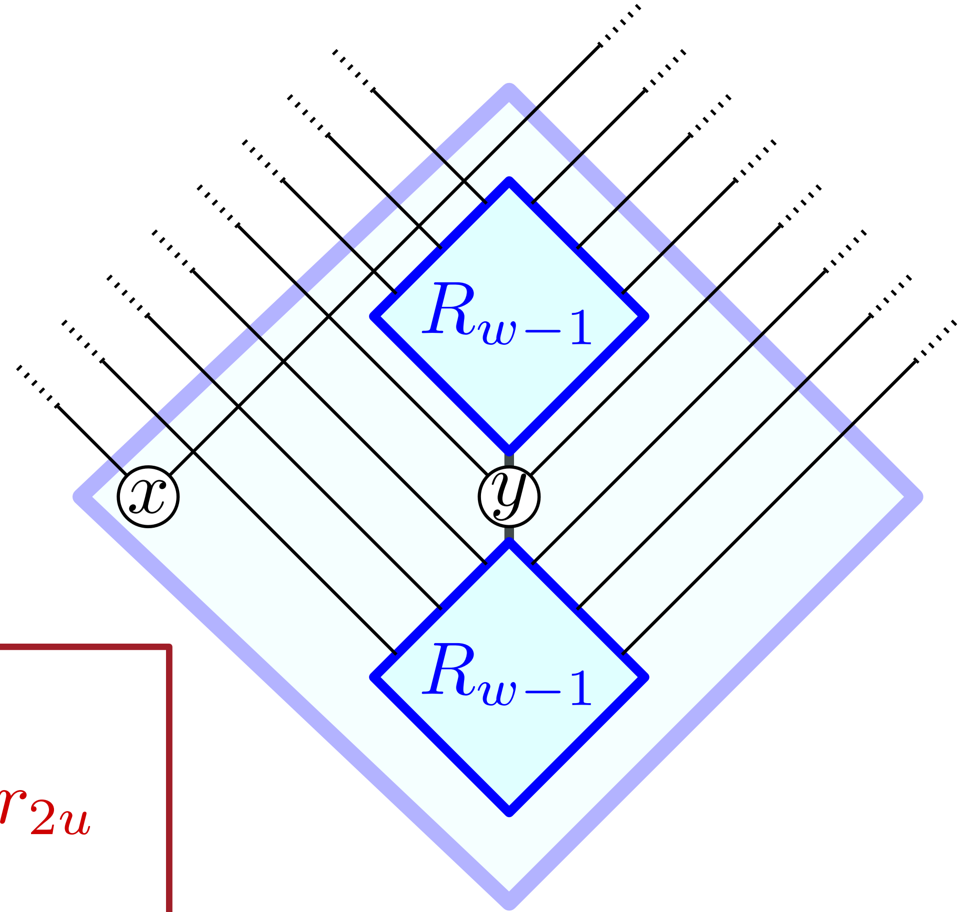
$$\Rightarrow \mathrm{qn}(P_w) \geq \sum_{u=1}^{w/2} \sqrt{2u} = \Omega(w^{3/2})$$

first attempt for R_w





the actual construction of R_w



$$\mathrm{qn}(P_w) \geq \mathrm{qn}(P_{w-2}) + r_{w-2} \geq \sum_{u=1}^{w/2} r_{2u}$$

- Erdős-Szekeres gives $r_w \geq \sqrt{w}$

$$\Rightarrow \mathrm{qn}(P_w) \geq \sum_{u=1}^{w/2} \sqrt{2u} = \Omega(w^{3/2})$$

Claim: $\forall L\colon \mathrm{width}(L \cap L_1) + \mathrm{width}(L \cap L_2) \geq w + 1$

$$\Rightarrow \mathrm{qn}(P_w) \geq \sum_{u=1}^{w/2} \frac{u+1}{2} \geq \frac{1}{8} w^2$$

summary / state of the art

Heath, Pemmaraju (1997)

$\forall w \ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

Alam et al. (GD 2020)

$\forall P \qquad \mathrm{qn}(P_w) \leq (w-1)^2 + 1$

Our Result.

$\forall w \geq 3 \ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

open problems

Find smallest c s.t.

$\forall w \ \forall P_w \qquad \mathrm{qn}(P_w) \leq c \cdot w^2$

summary / state of the art

Heath, Pemmaraju (1997)

$\forall w \ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

Alam et al. (GD 2020)

$\forall P \qquad \mathrm{qn}(P_w) \leq (w-1)^2 + 1$

Our Result.

$\forall w \geq 3 \ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

open problems

Find smallest c s.t.

$\forall w \ \forall P_w \qquad \mathrm{qn}(P_w) \leq c \cdot w^2$

Is it true that $\mathrm{qn}(P_w) \leq w$

- if $\dim(P_w) \leq 2$?

summary / state of the art

Heath, Pemmaraju (1997)

$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

Alam et al. (GD 2020)

$\forall P \qquad \mathrm{qn}(P_w) \leq (w-1)^2 + 1$

Our Result.

$\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

open problems

Find smallest c s.t.

$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq c \cdot w^2$

Is it true that $\mathrm{qn}(P_w) \leq w$

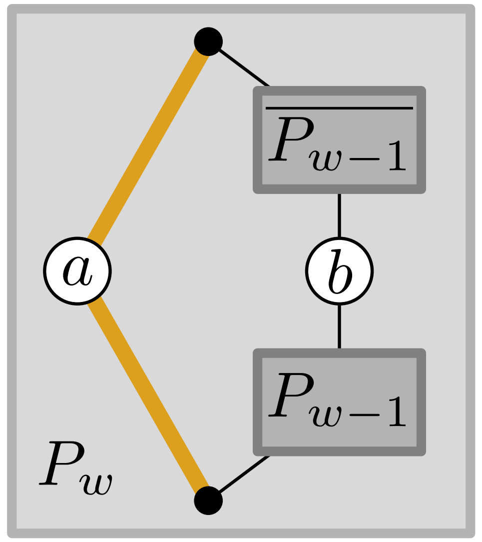
- if $\dim(P_w) \leq 2$?
- if P_w is planar?

planar posets

Knauer, Micek, U. (GD 2018)

$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq 3w - 2$

$\forall w\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w$



summary / state of the art

Heath, Pemmaraju (1997)

$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq w^2$

Alam et al. (GD 2020)

$\forall P \qquad \mathrm{qn}(P_w) \leq (w-1)^2 + 1$

Our Result.

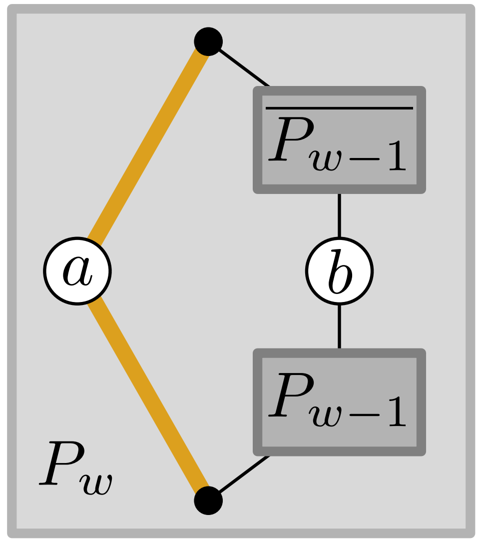
$\forall w \geq 3\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w^2/8$

planar posets

Knauer, Micek, U. (GD 2018)

$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq 3w - 2$

$\forall w\ \exists P_w \qquad \mathrm{qn}(P_w) \geq w$



open problems

Find smallest c s.t.
$\forall w\ \forall P_w \qquad \mathrm{qn}(P_w) \leq c \cdot w^2$

Is it true that $\mathrm{qn}(P_w) \leq w$
- if $\dim(P_w) \leq 2$?
- if P_w is planar?

Conjecture (Heath, Pemmaraju 1997).
Is it true that $\mathrm{qn}(P) \leq \sqrt{|P|}$
if P is planar?

Thank You!