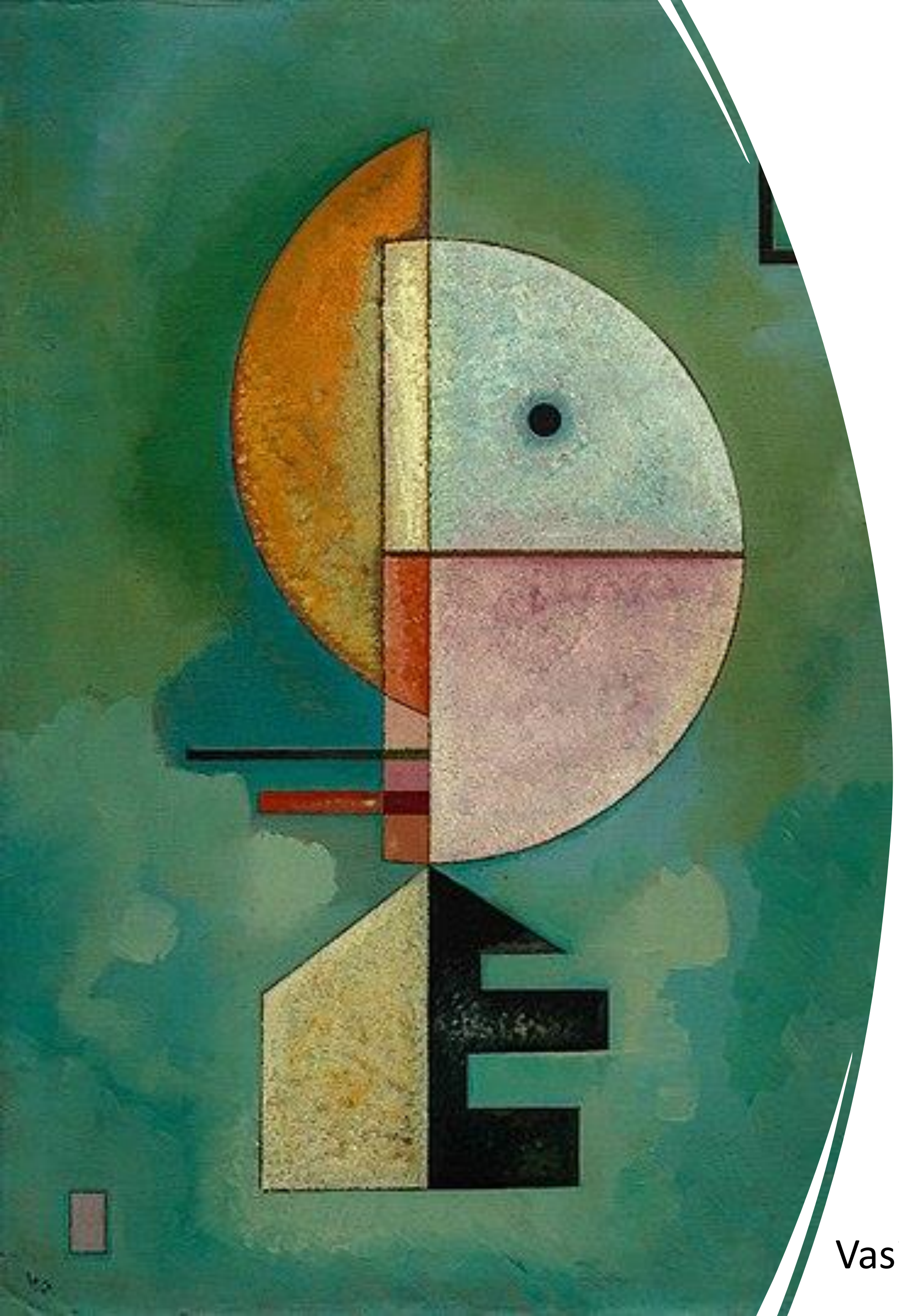
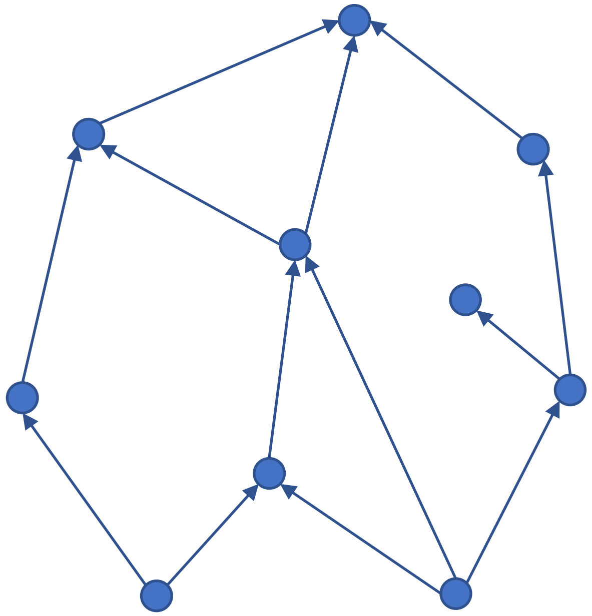


Quasi-upward Planar Drawings with Minimum Curve Complexity

Carla Binucci, Emilio Di Giacomo, Giuseppe Liotta, Alessandra Tappini

Vasily Kandinsky, **Upward**, Peggy Guggenheim Collection, Venice

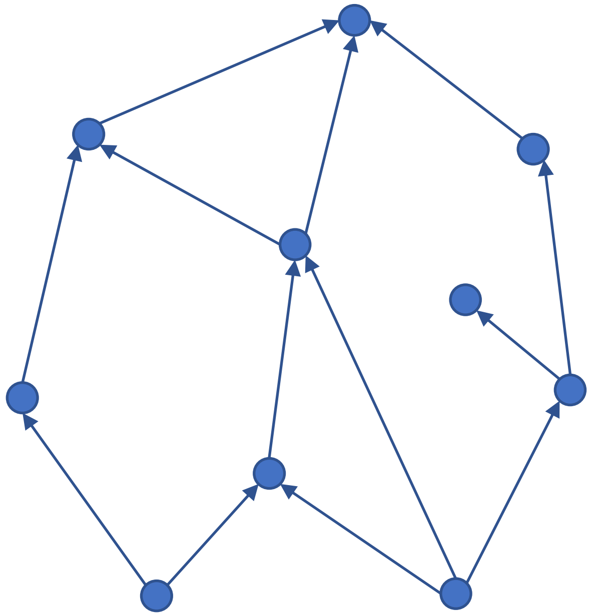
Upward planar drawings



An upward planar drawing
of a digraph G

- No edges cross each other.
- All the edges are represented by curves monotonically increasing in the vertical direction.

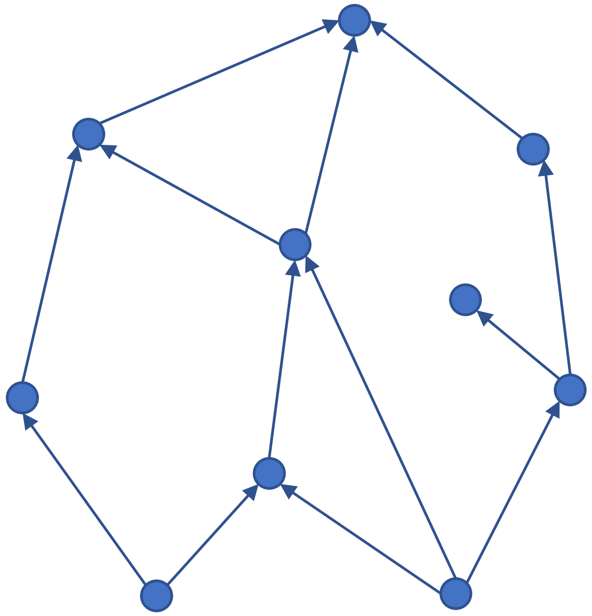
Upward planar drawings



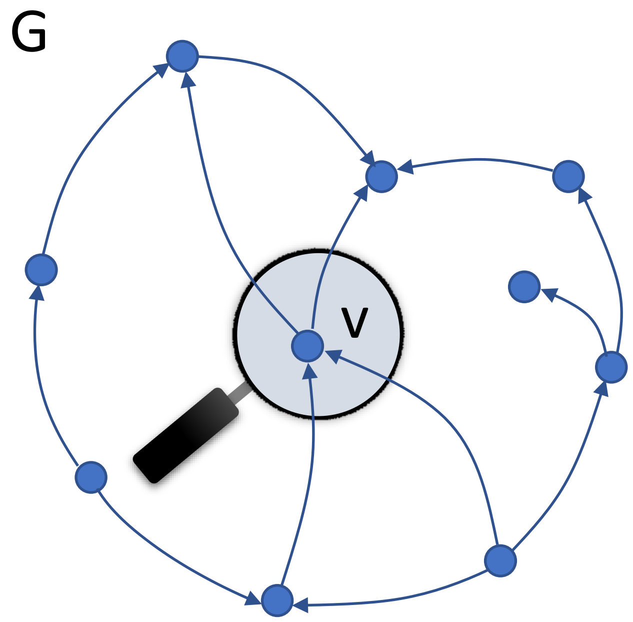
An upward planar drawing
of a digraph G

Acyclicity and bimodality are necessary conditions for upward planarity.

Upward planar drawings - bimodality



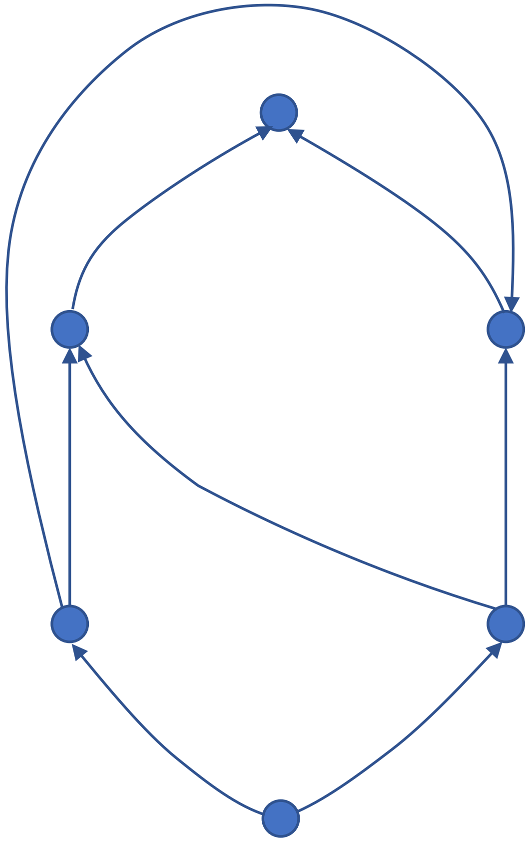
An upward planar drawing
of a digraph G



Each vertex v is *bimodal*: the incoming and outgoing edges of v never alternate around v

Upward planar drawings

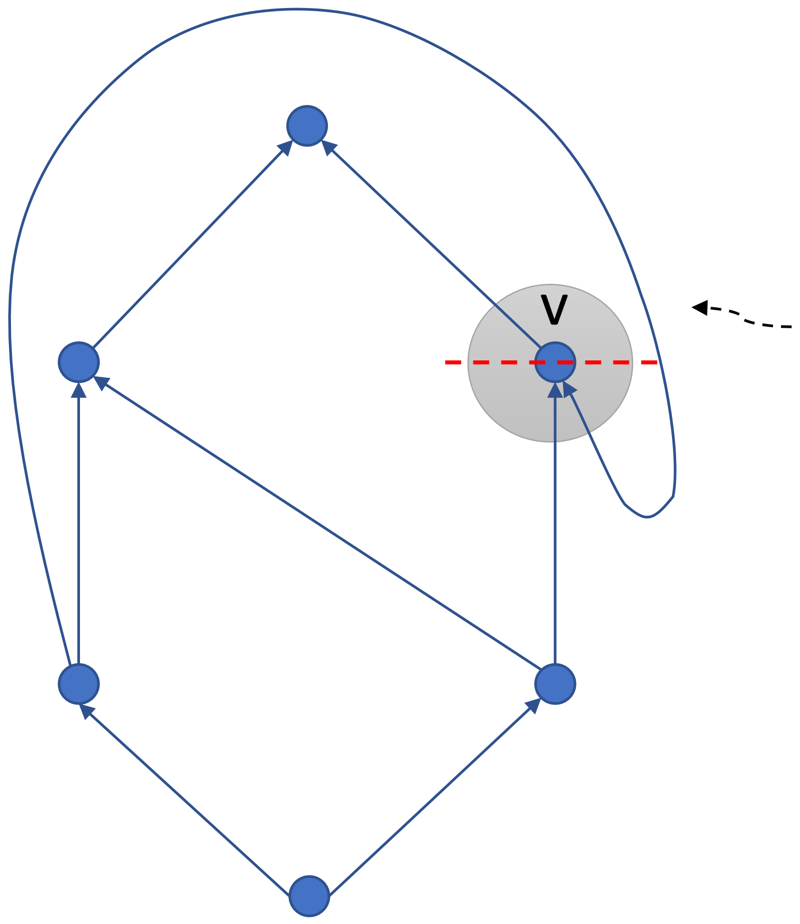
Acyclicity and bimodality are **not sufficient conditions** for upward planarity.



A bimodal acyclic planar digraph G
that is not upward planar.

Quasi-upward planar drawings

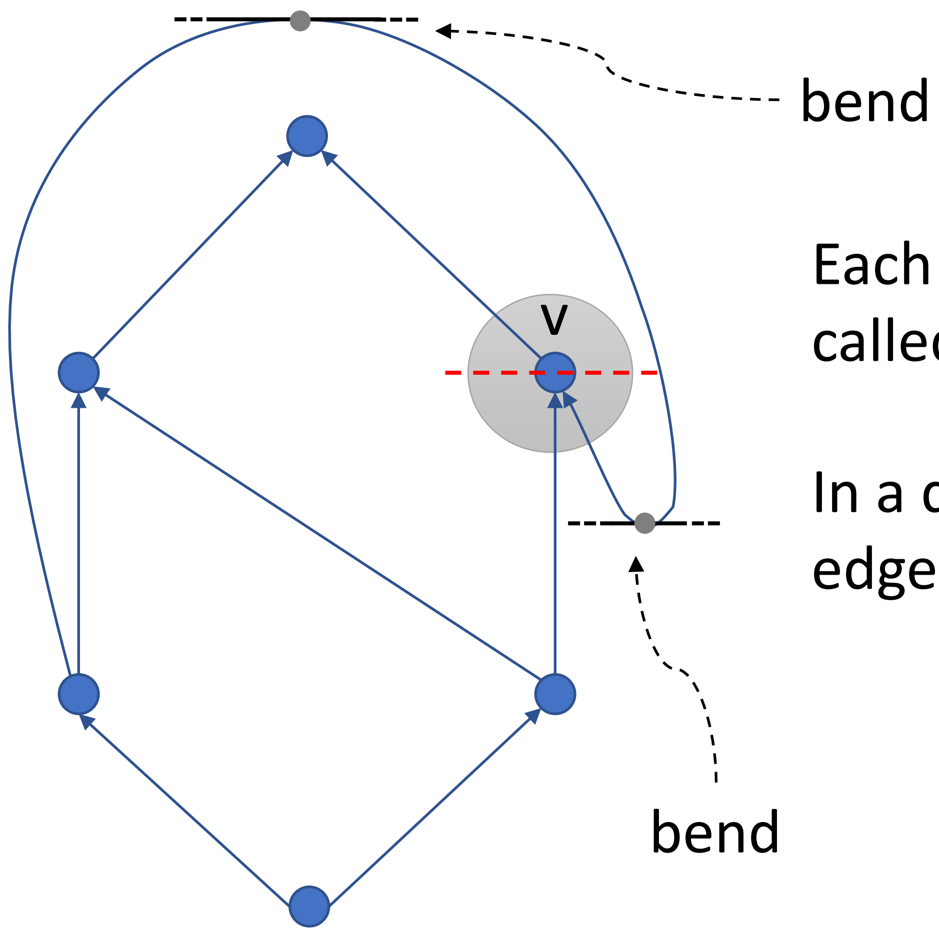
A quasi-upward planar drawing Γ of G



For each vertex v, all the incoming edges enter v from “below” and all the outgoing edges leave v from “above".

Quasi-upward planar drawings

A quasi-upward planar drawing Γ of G



Each point of horizontal tangency is called a *bend*.

In a quasi-upward planar drawing each edge has an **even** number of bends.

Upward planarity - known results

- Upward planarity testing is polynomial time solvable in the fixed embedding setting.

  [Bertolazzi, Di Battista, Liotta, Mannino - Upward drawings of triconnected digraphs. Algorithmica. 1994]

- Upward planarity testing is NP-hard in the variable embedding setting.

  [Garg and Tamassia - On the computational complexity of upward and rectilinear planarity testing. SIAM J. Comput. 2001]

Quasi-upward planarity - known results

Bertolazzi et al., provide the following results:

- Every plane bimodal digraph admits a quasi-upward planar drawing.
- In the fixed embedding setting, they give an $O(n^2)$-time algorithm to compute a quasi-upward planar drawing with the minimum number of bends. The algorithm uses a min-cost flow technique.
- In the variable embedding setting, they provide a branch and bound algorithm for computing a quasi-upward planar drawing with the minimum number of bends of a biconnected digraph has been provided.

[Bertolazzi, Di Battista, Didimo - Quasi-Upward planarity. Algorithmica. 2002]

Our problem

We study the problem of computing quasi-upward planar drawings of bimodal plane digraphs with minimum *curve complexity* (i.e. the maximum number of bends along any edge of the drawing).

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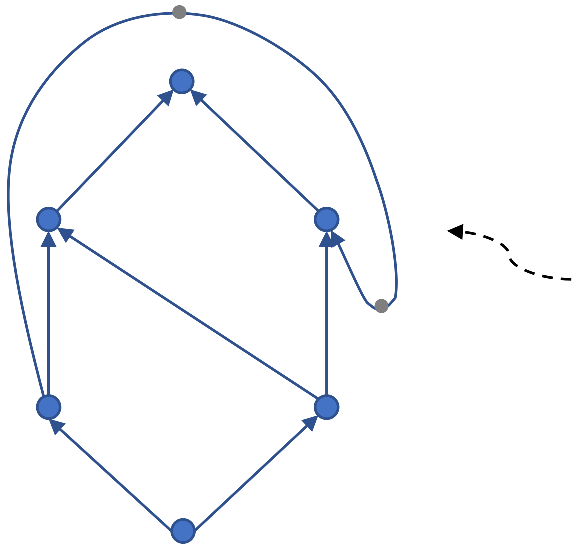


curve complexity: 2

Our contribution (i)

i. We show that every bimodal plane digraph admits an embedding preserving quasi-upward planar drawing with curve complexity two.

Our contribution (ii)

ii. We provide an $\tilde{O}(m^{\frac{4}{3}})$-time algorithm to compute embedding-preserving quasi-upward planar drawings that minimize the curve complexity and that have the minimum number of bends when no edge can be bent more than twice.

Our contribution (iii)

iii. We show that for every $n \geq 39$ there exists a planar bimodal digraph with n vertices whose bend-minimum quasi-upward planar drawings have $\Omega(n)$ bends on a single edge.

This bound holds even in the variable embedding setting.

Analogous results in the orthogonal setting

i. We show that every bimodal plane digraph admits an embedding preserving quasi-upward planar drawing with curve complexity two.

> ➢ In the orthogonal setting, Biedl and Kant prove that every plane graph of degree at most four admits an orthogonal drawing with curve complexity two.
>
> [Biedl, Kant - A better heuristic for orthogonal graph drawings. Comput. Geom. (1998)]

Analogous results in the orthogonal setting

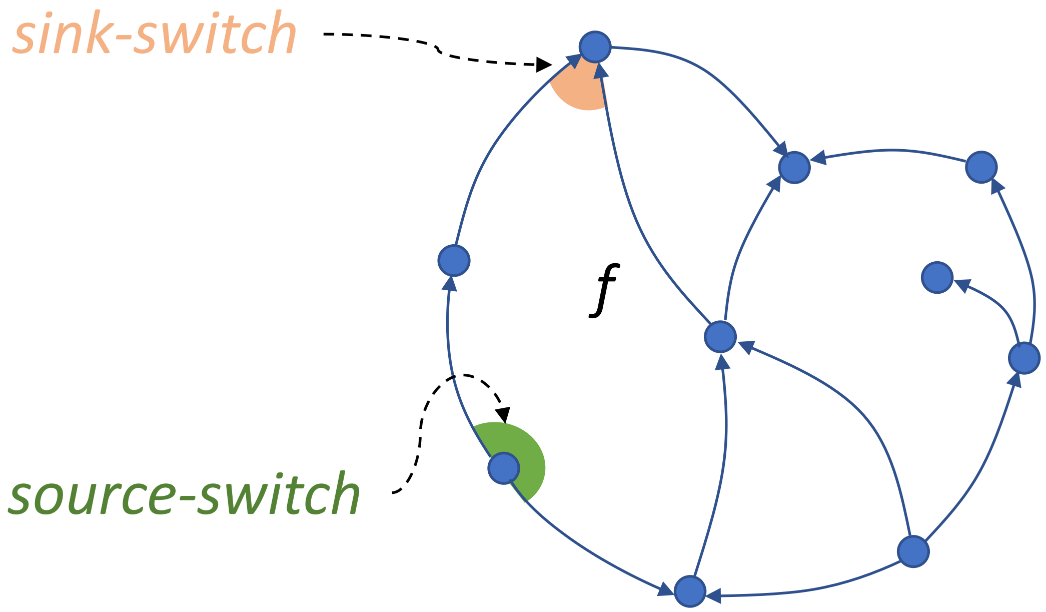
iii. We show that for every $n \geq 39$ there exists a planar bimodal digraph with n vertices whose bend-minimum quasi-upward planar drawings have $\Omega(n)$ bends on a single edge.

This bound holds even in the variable embedding setting.

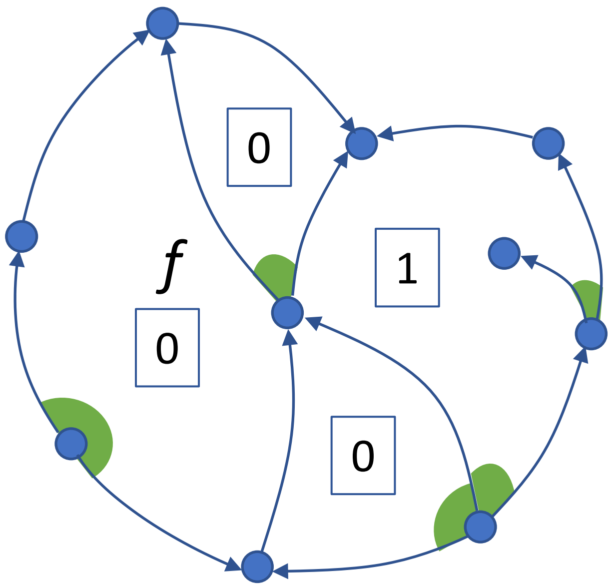
- In the orthogonal setting, Tamassia et al. show a similar lower bound on the curve complexity of bend-minimum planar orthogonal representations.
  [Tamassia, Tollis, Vitter - Lower bounds and parallel algorithms for planar orthogonal grid drawings. Third IEEE Symposium on Parallel and Distributed Processing. (1991)]

Some preliminary definitions

Bimodal planar digraphs – some definitions



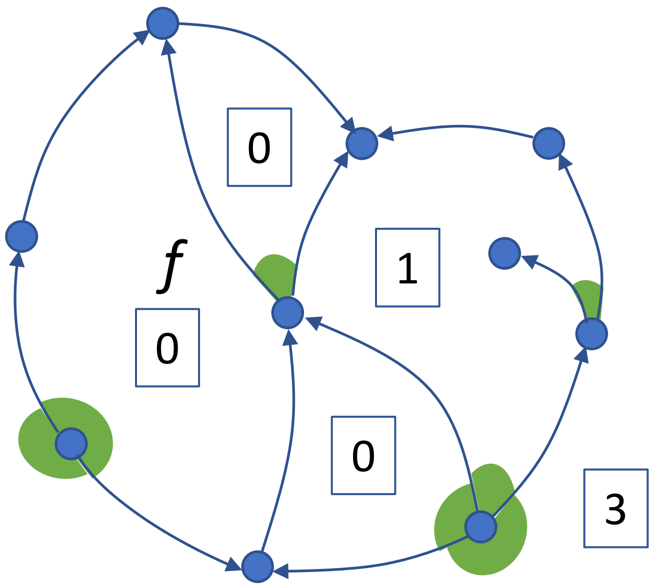
Bimodal planar digraphs – some definitions



Capacity of a face *f* :

source-switches – 1 , if *f* is internal

Bimodal planar digraphs – some definitions

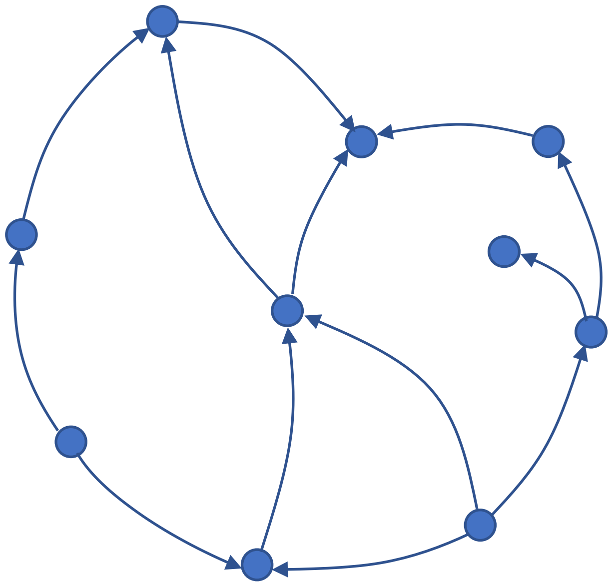


Capacity of a face *f* :

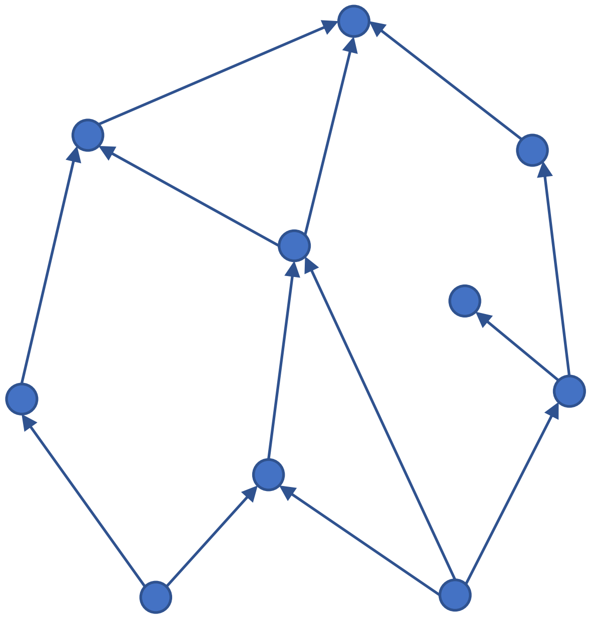
source-switches – 1 , if *f* is internal

source-switches + 1 , if *f* is external

Upward consistent assignment

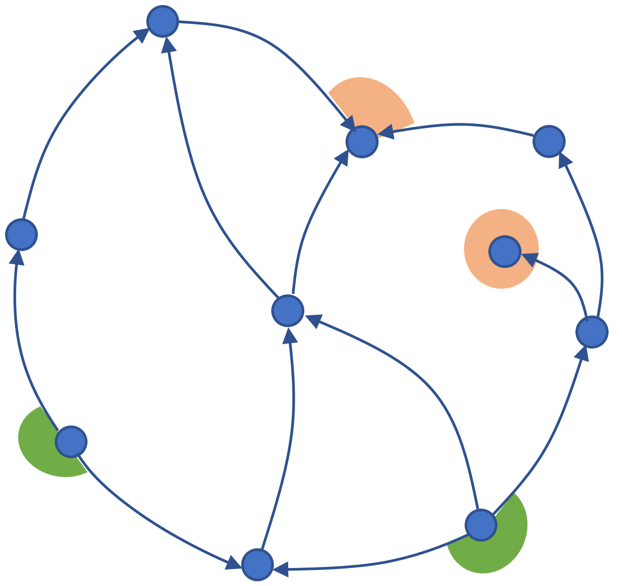


G



An upward planar drawing Γ of G

Upward consistent assignment



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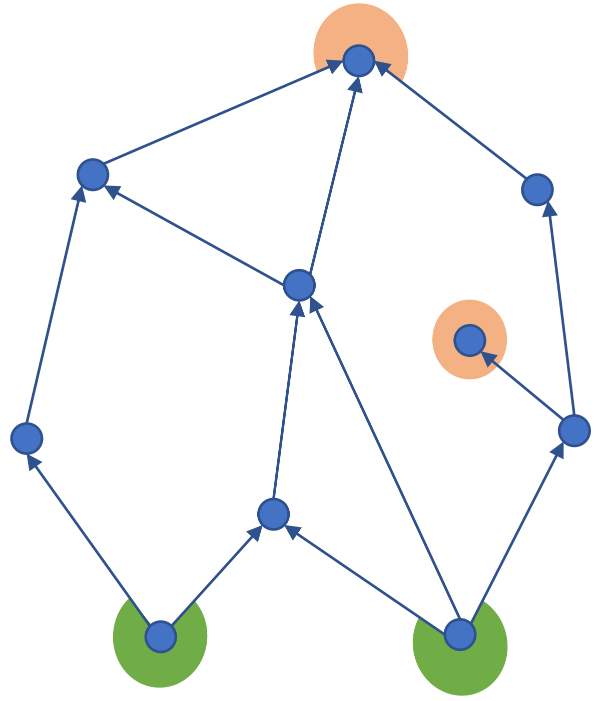

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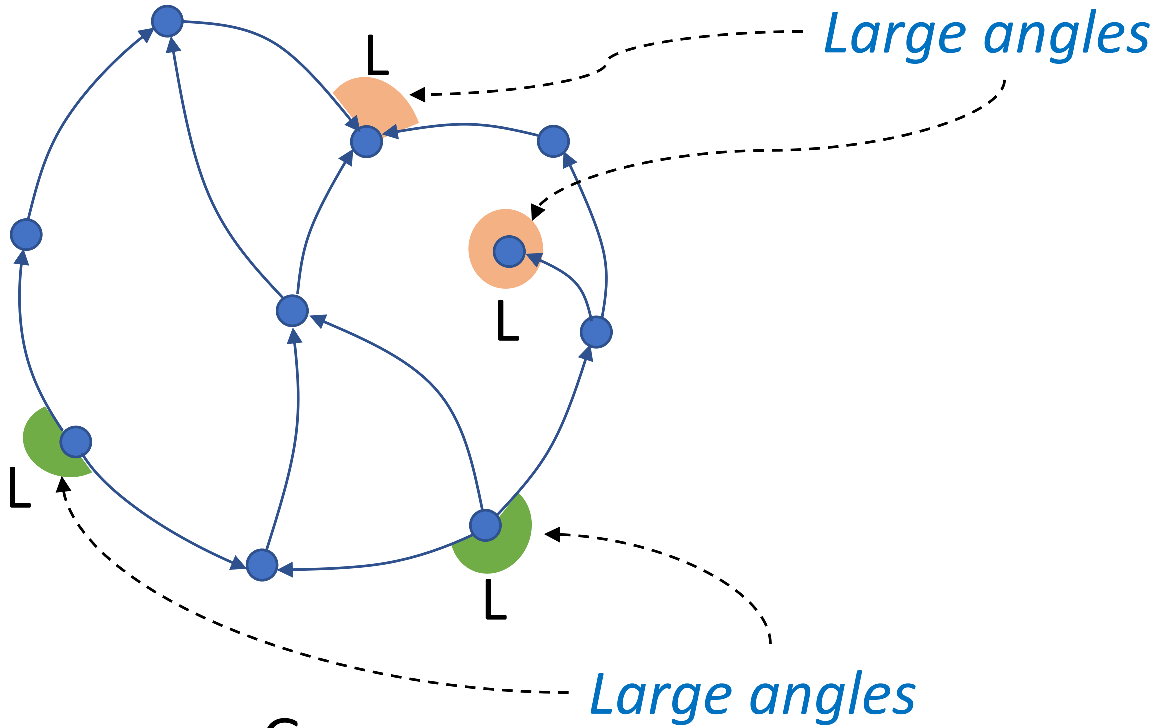
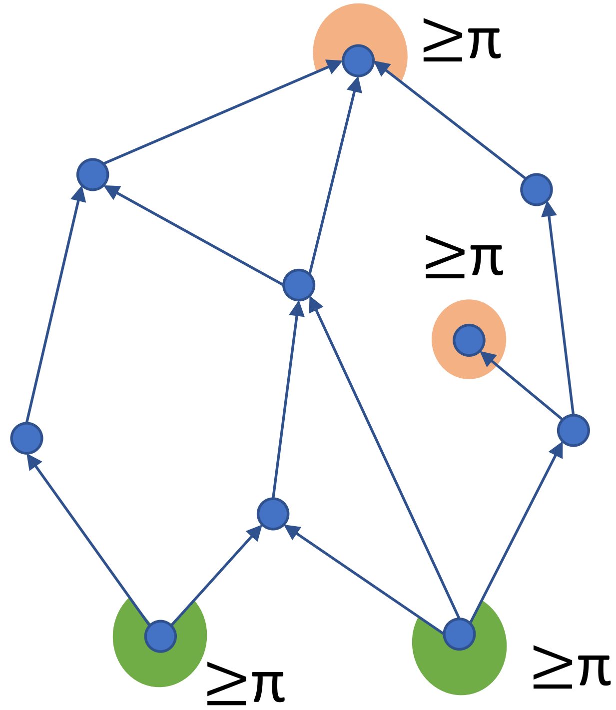


G



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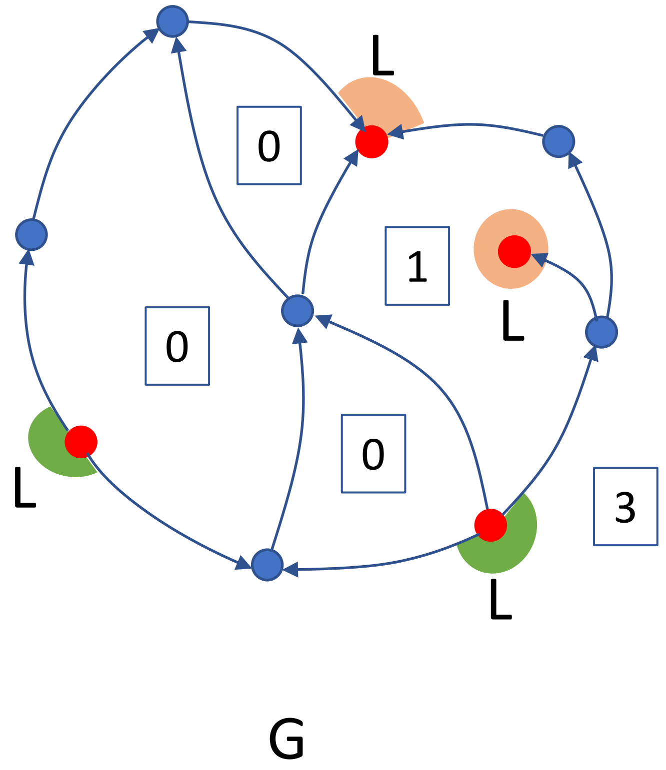
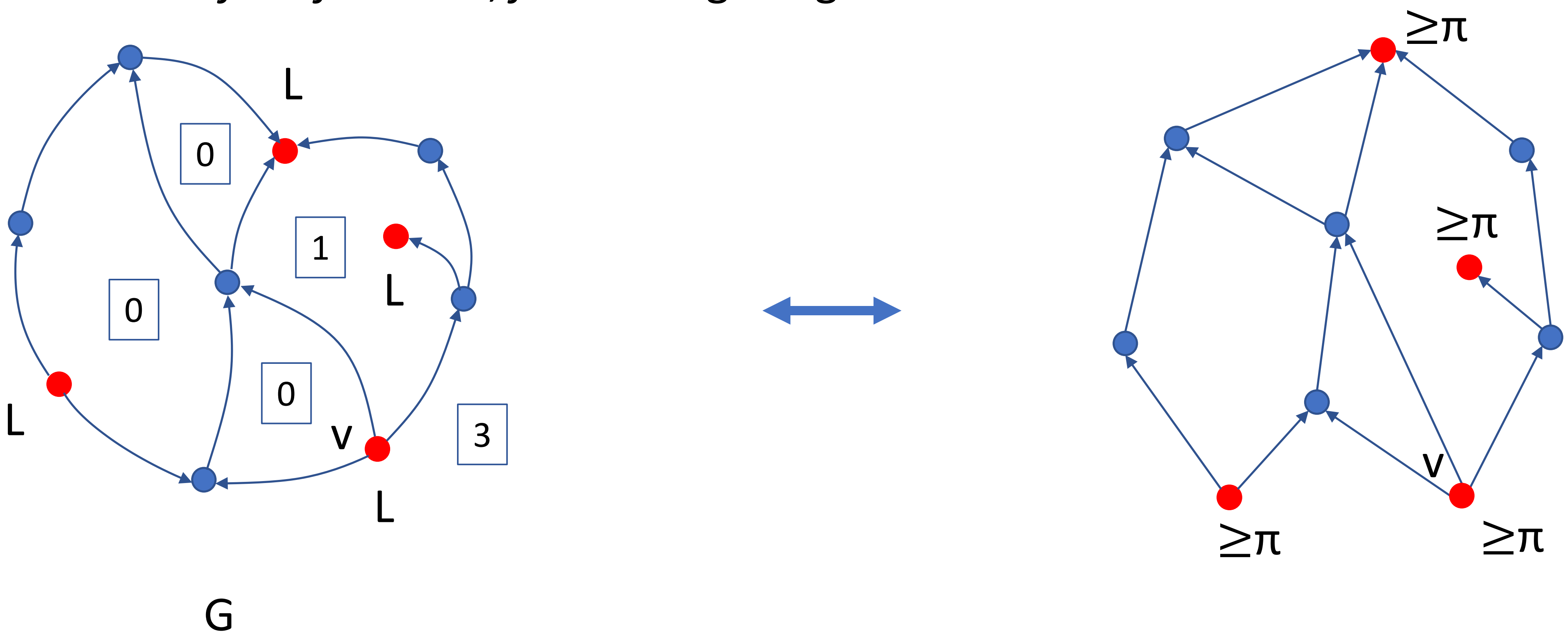


- A sink or a source vertex has exactly one L label on one of its angles.
- For a face, the number of large angles, is equal to its capacity.

Upward consistent assignment

Assigning an L label to an angle that a source or a sink *v* forms in *f* corresponds to *assign that source or sink to face f*. That is, *f* has a large angle at v.



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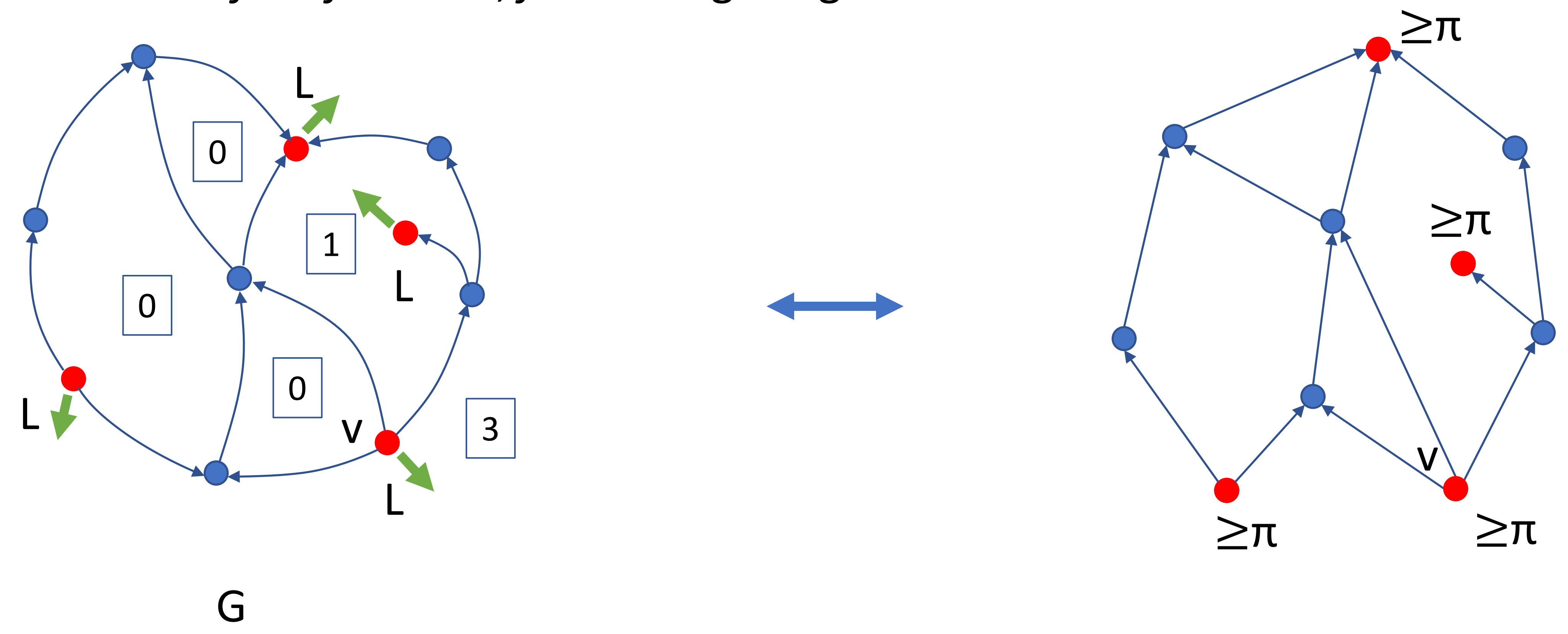
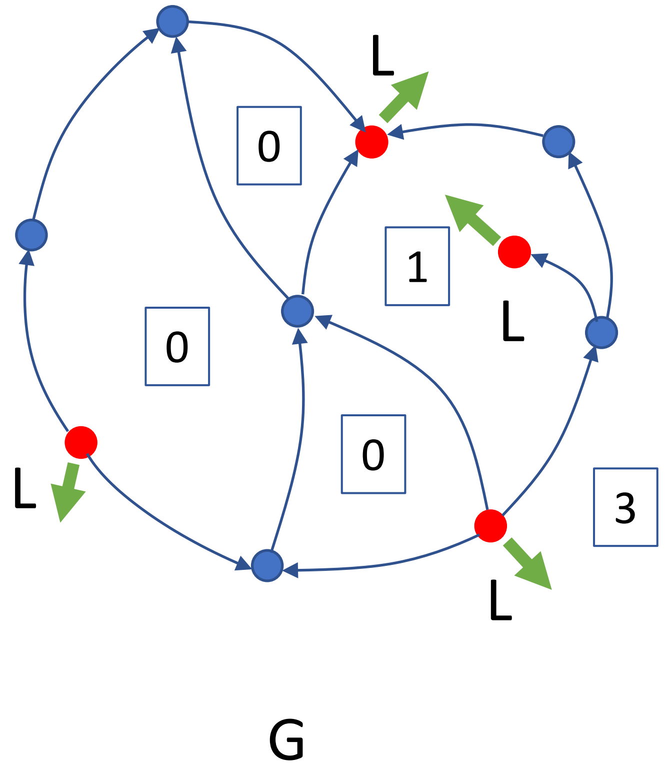


Upward consistent assignment



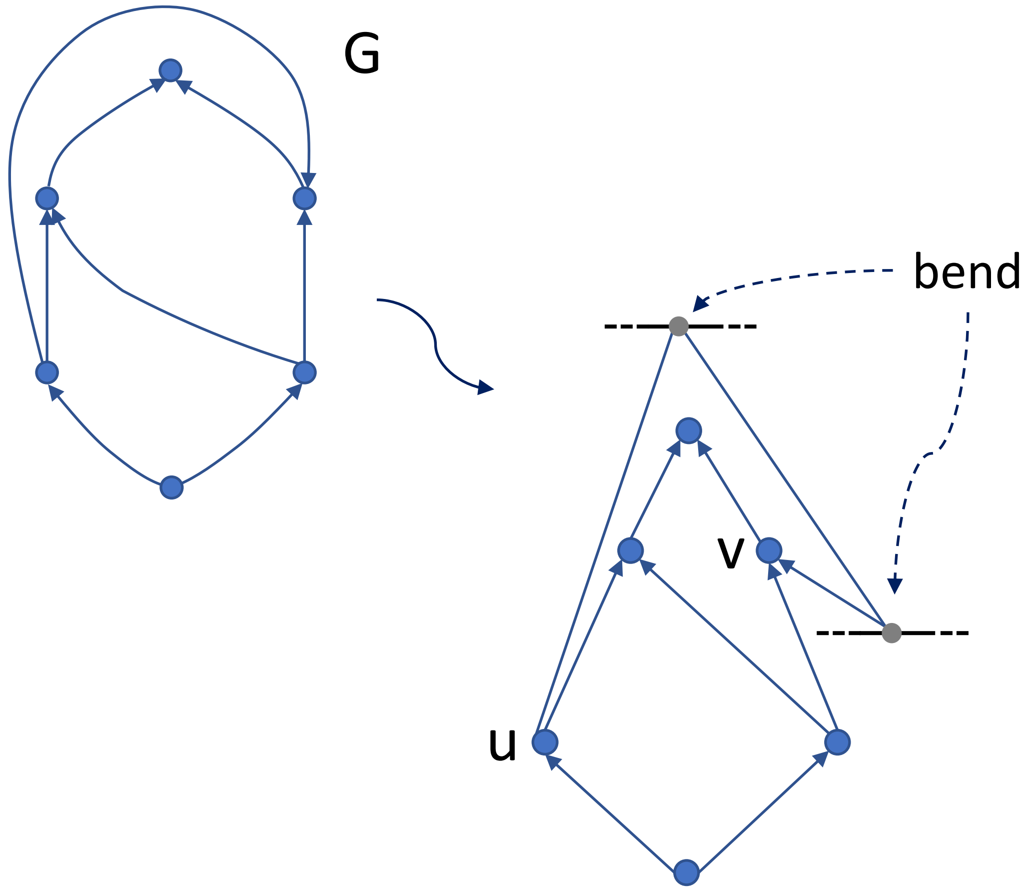
An *upward consistent assignment* is an assignment of the source and sink vertices of G to its faces such that:

(i) Each source or sink is assigned to exactly one of its incident faces.

(ii) For each face *f*, the number of source and sink vertices assigned to *f* is equal to the capacity of *f*.

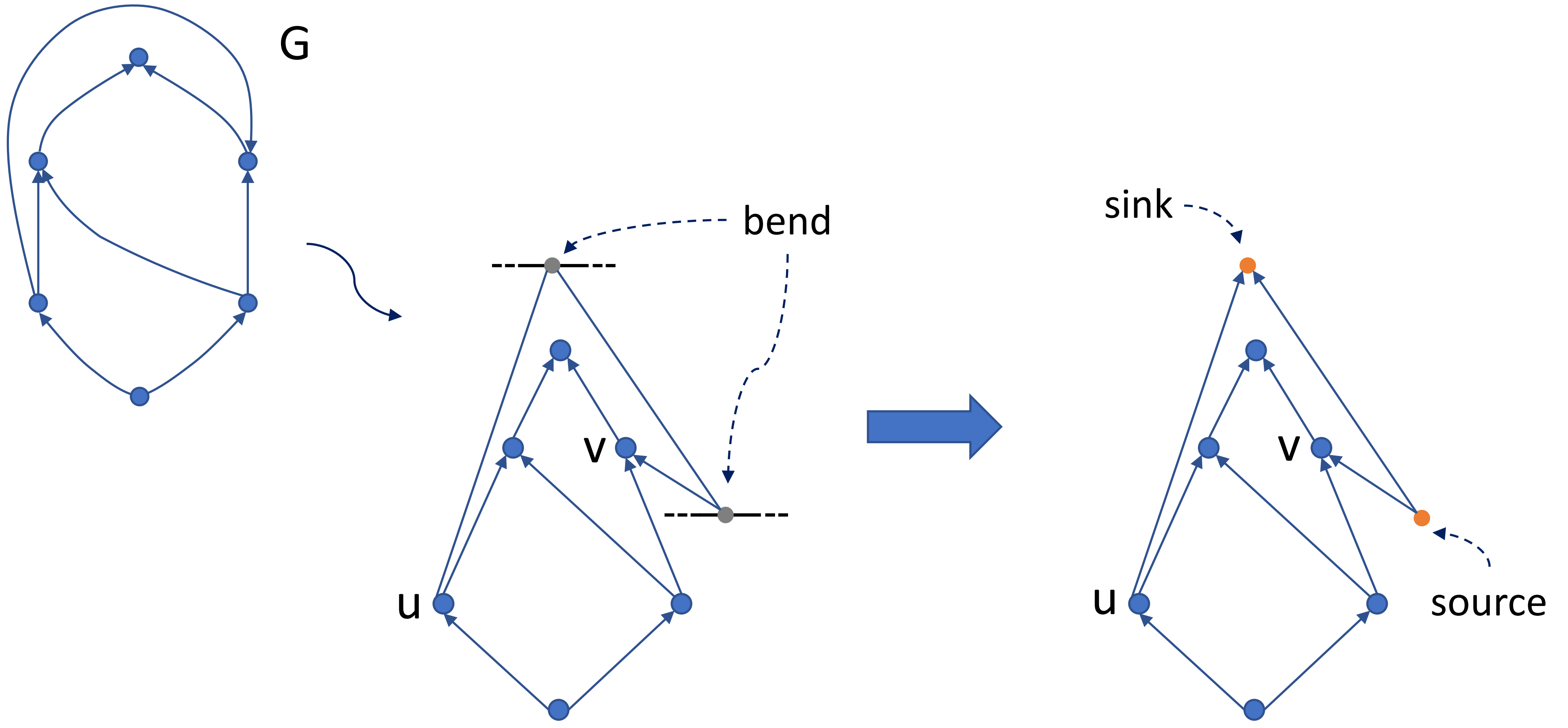
Every bimodal plane digraph admits a quasi-upward planar drawing with curve complexity two.

A preliminary observation



A quasi-upward
planar drawing Γ of G

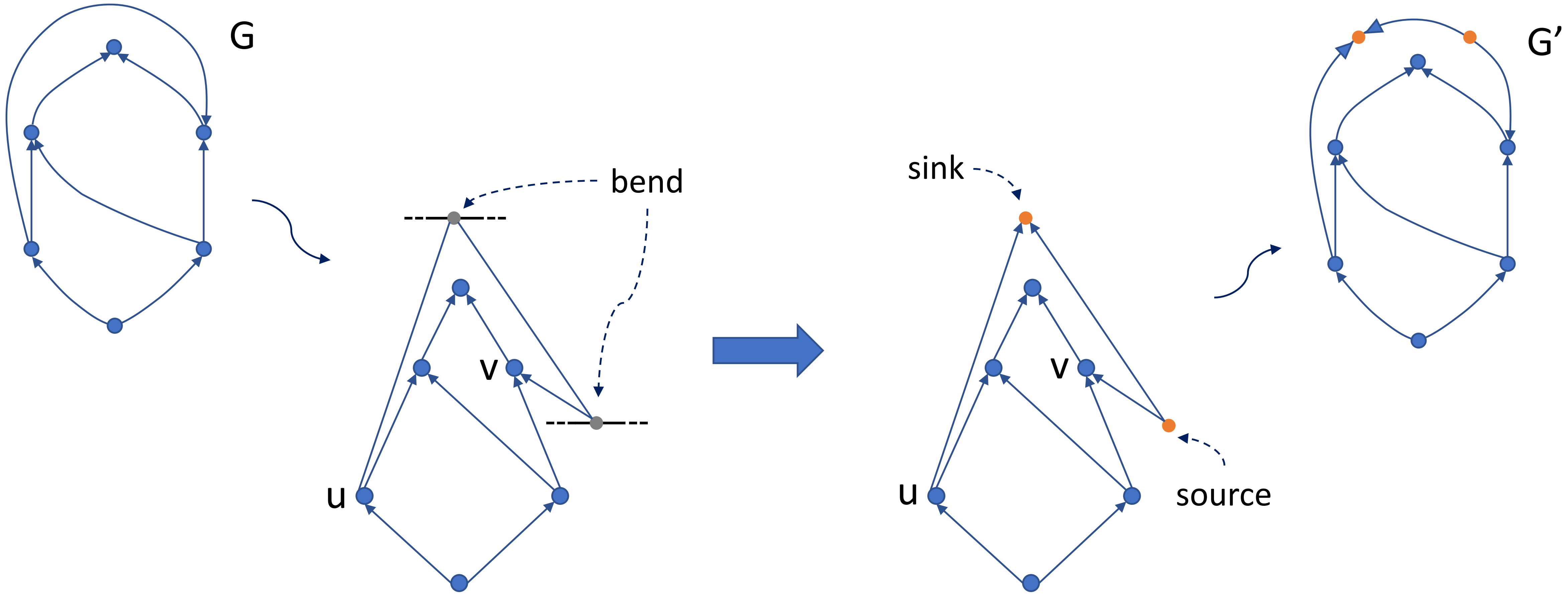
A preliminary observation



A quasi-upward planar drawing Γ of G

The *underlying* upward planar drawing Γ’ of Γ

A preliminary observation

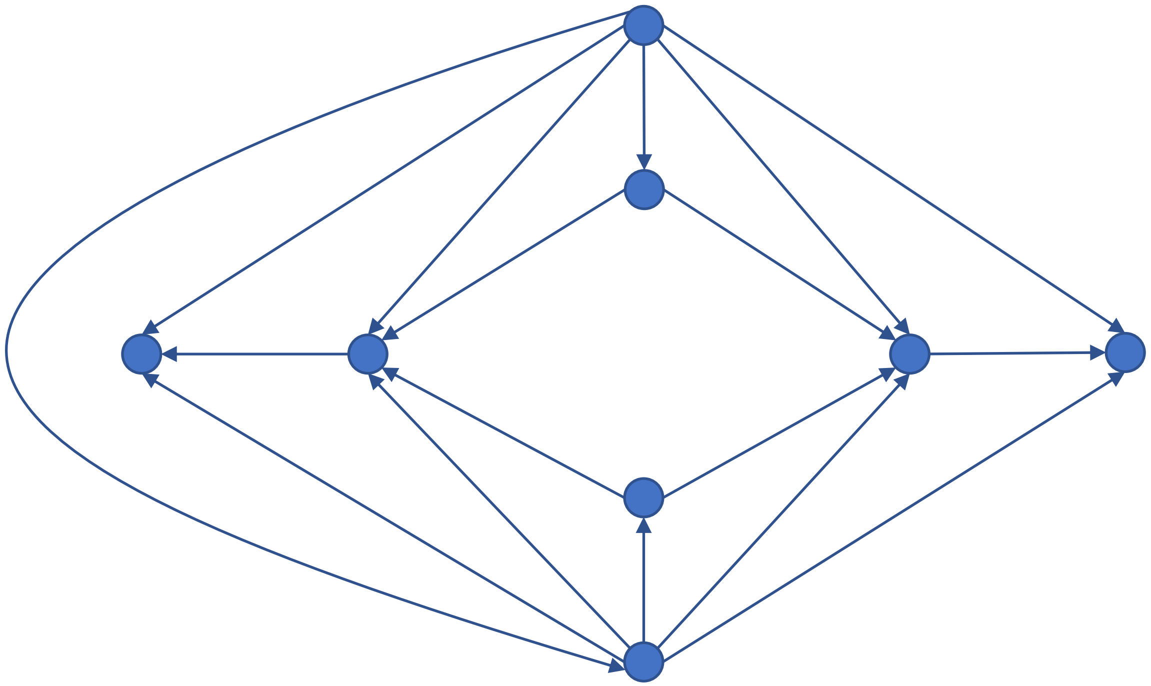


A quasi-upward planar drawing Γ of G

The *underlying* upward planar drawing Γ’ of Γ

Proof idea

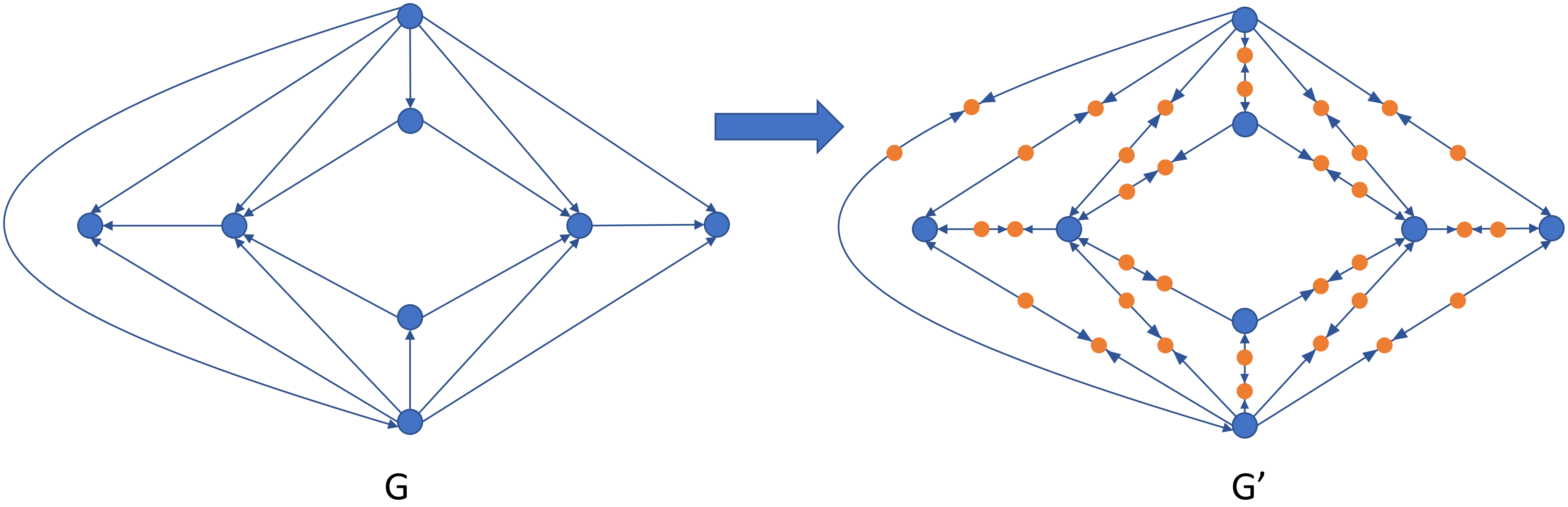
We consider a bimodal plane digraph G (possibly with multiple edges and not acyclic).



G

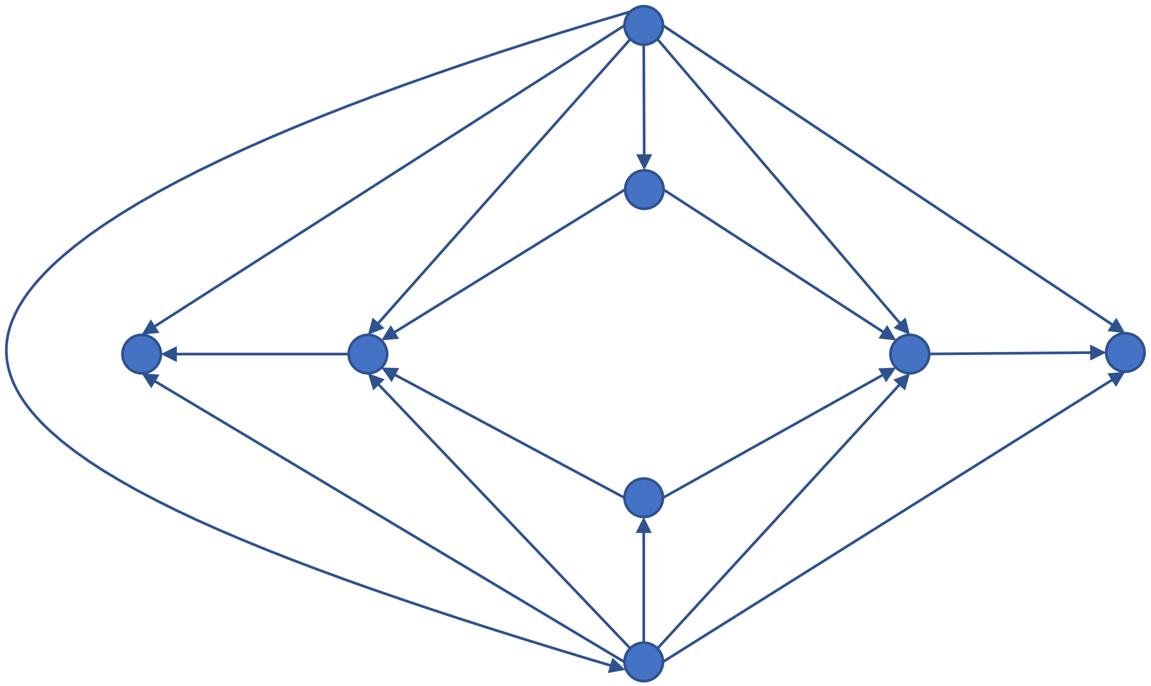
Proof idea

We subdivide each edge of G by inserting two subdivision vertices (one sink and one source).

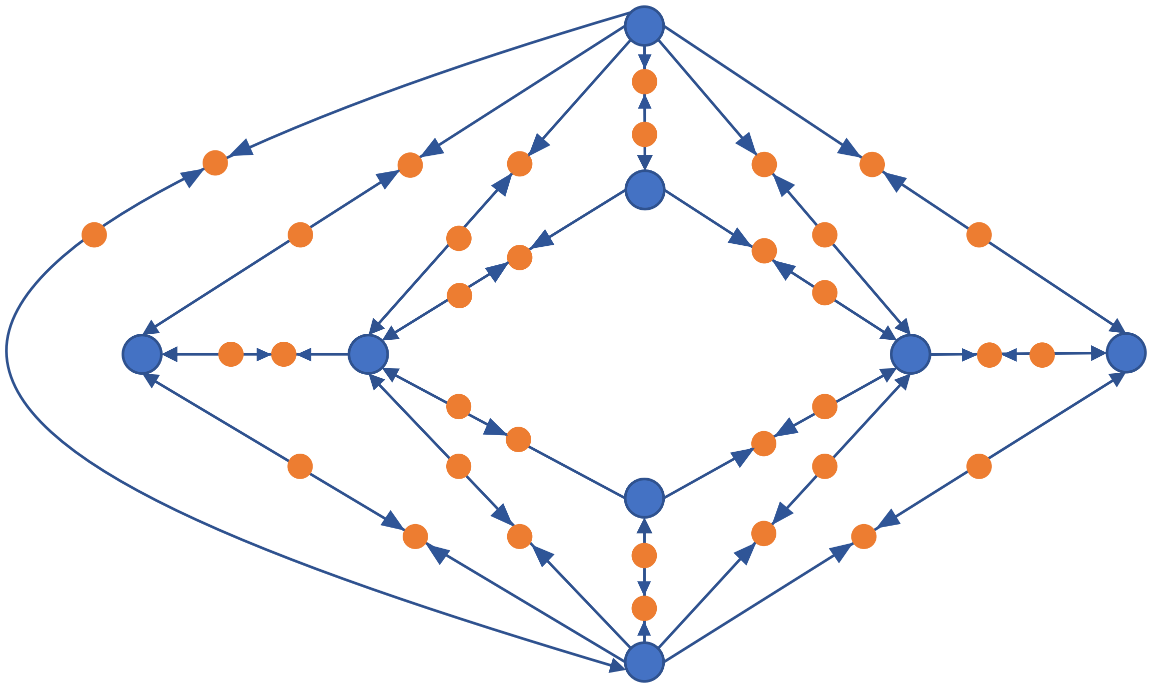


Proof idea

We prove that G’ is upward planar



G

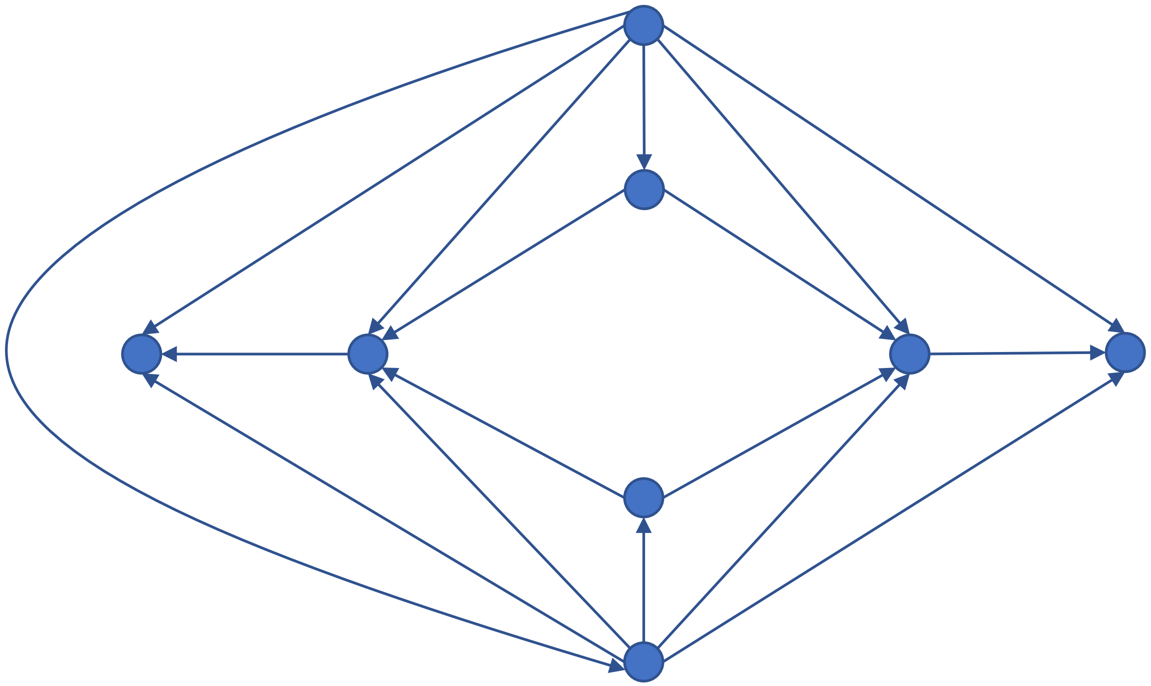


G’

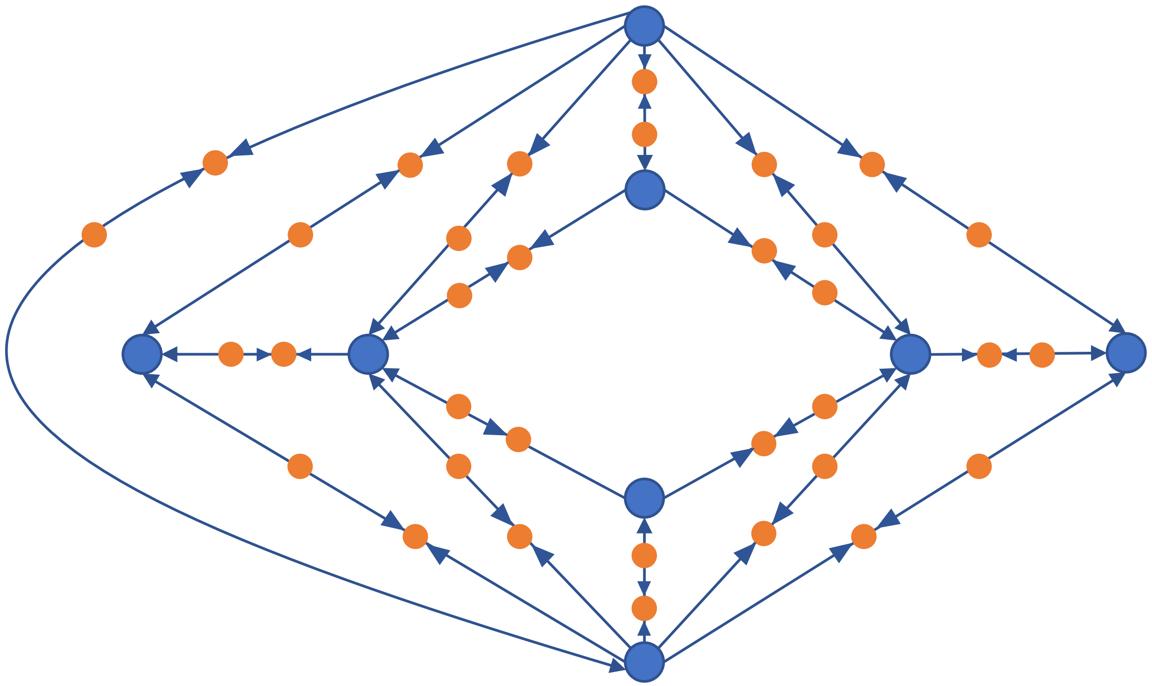
Proof idea

G admits a quasi-upward planar drawing with curve complexity 2.

We prove that G' is upward planar



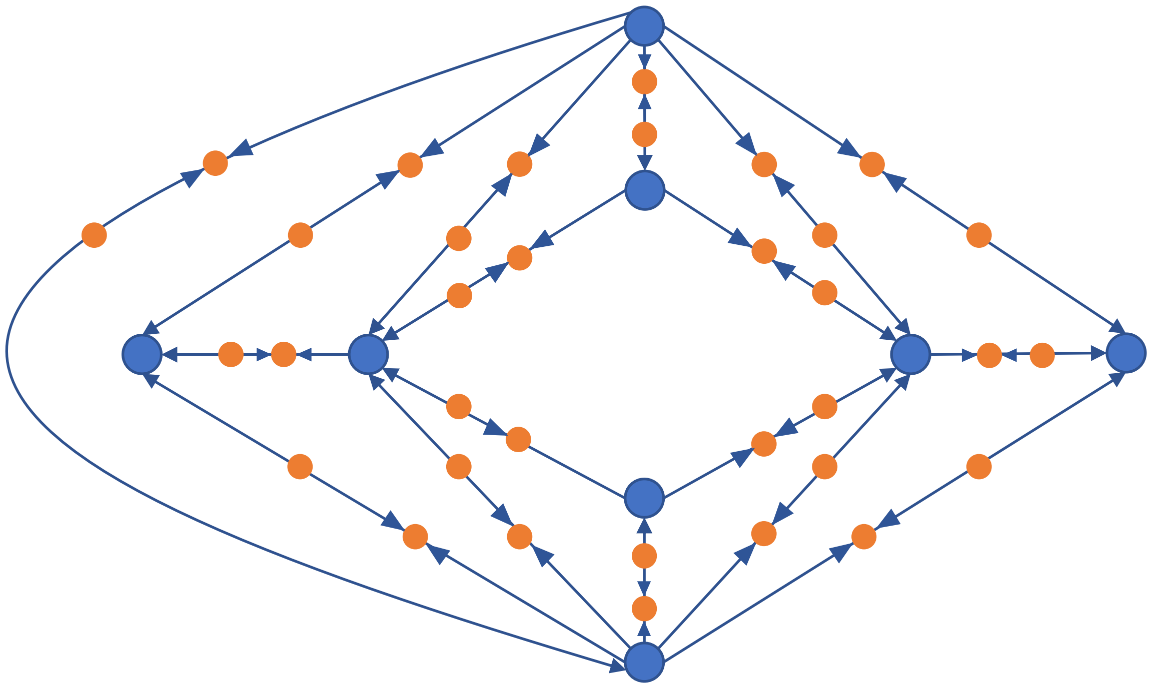
G



G'

Proof technique

We show that G’ admits an upward consistent assignment.



G’

Proof technique

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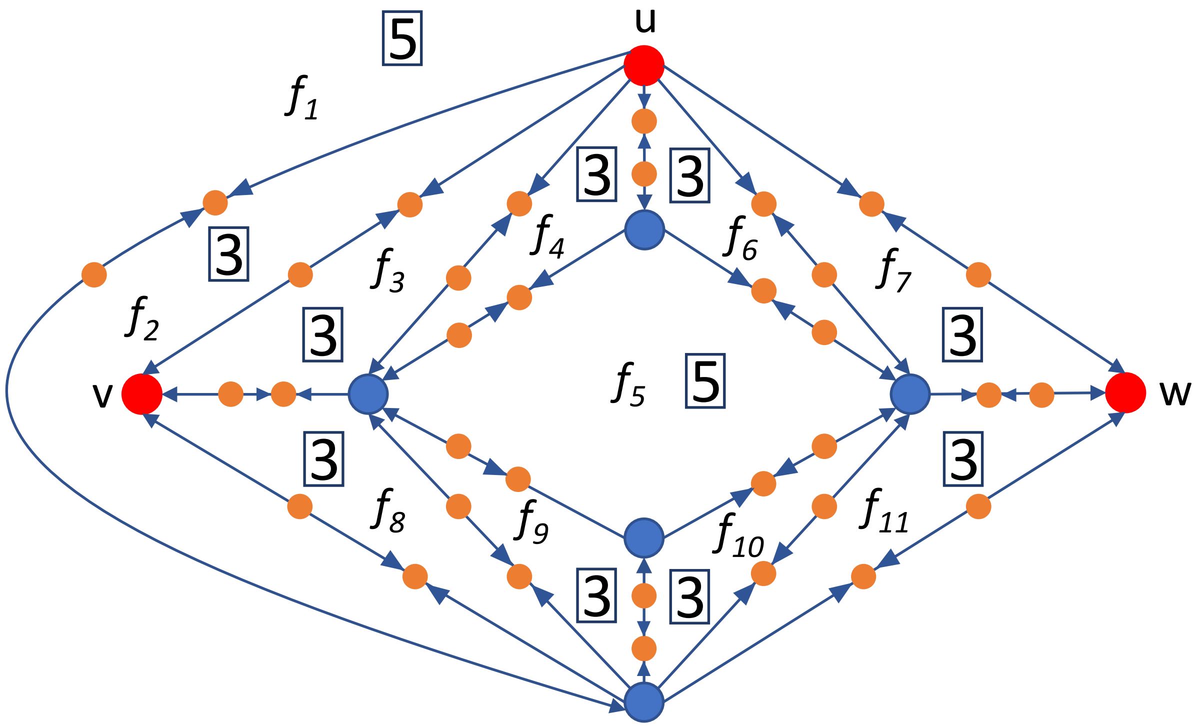
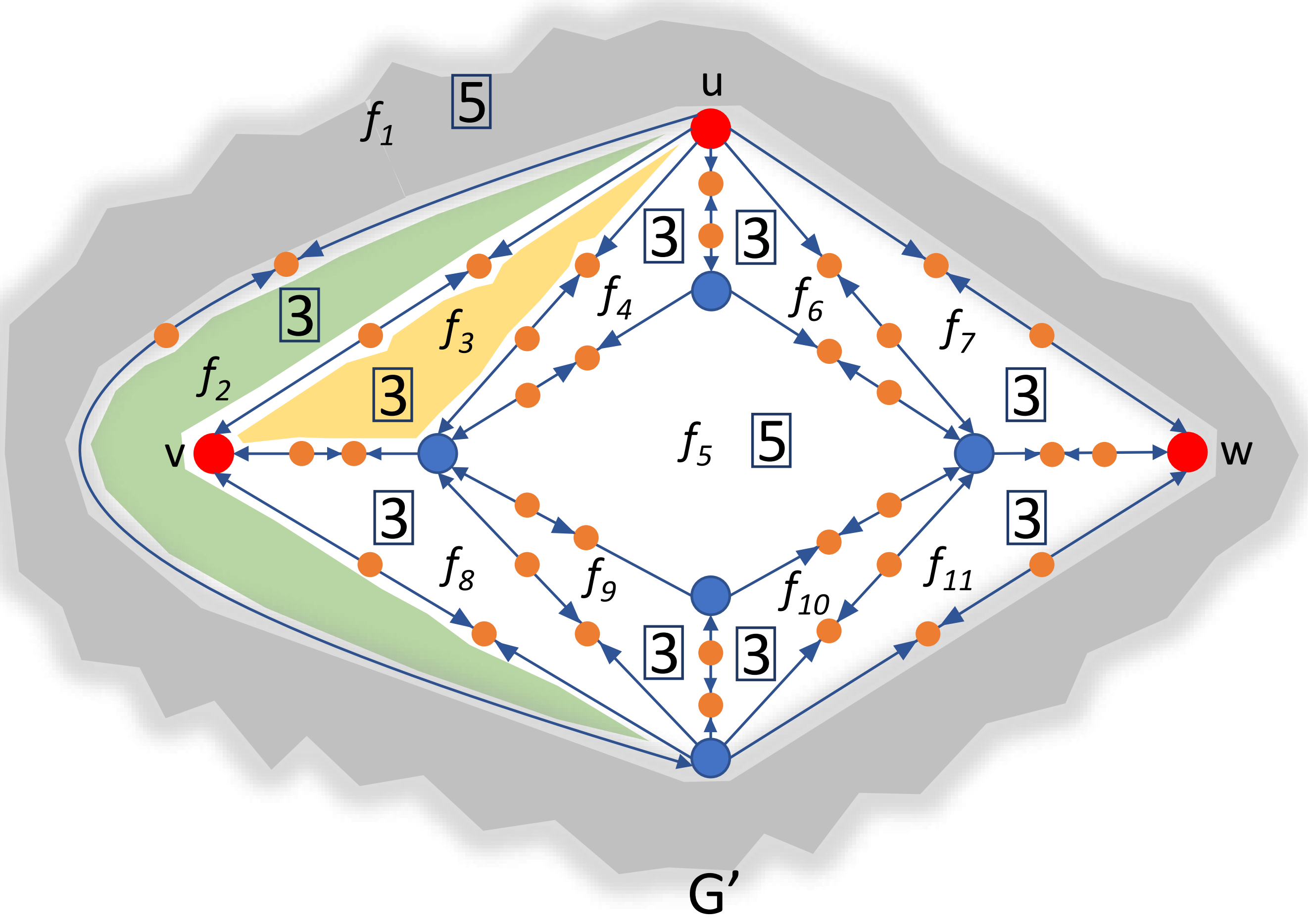
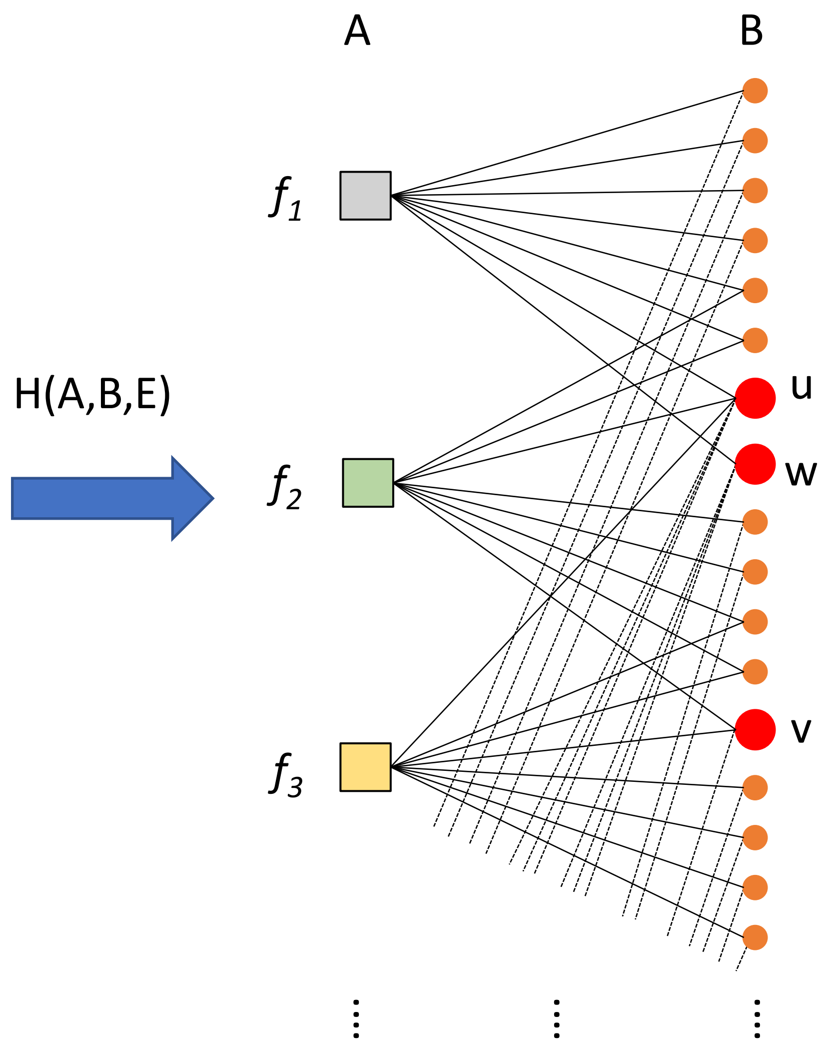


G’

A bipartite description





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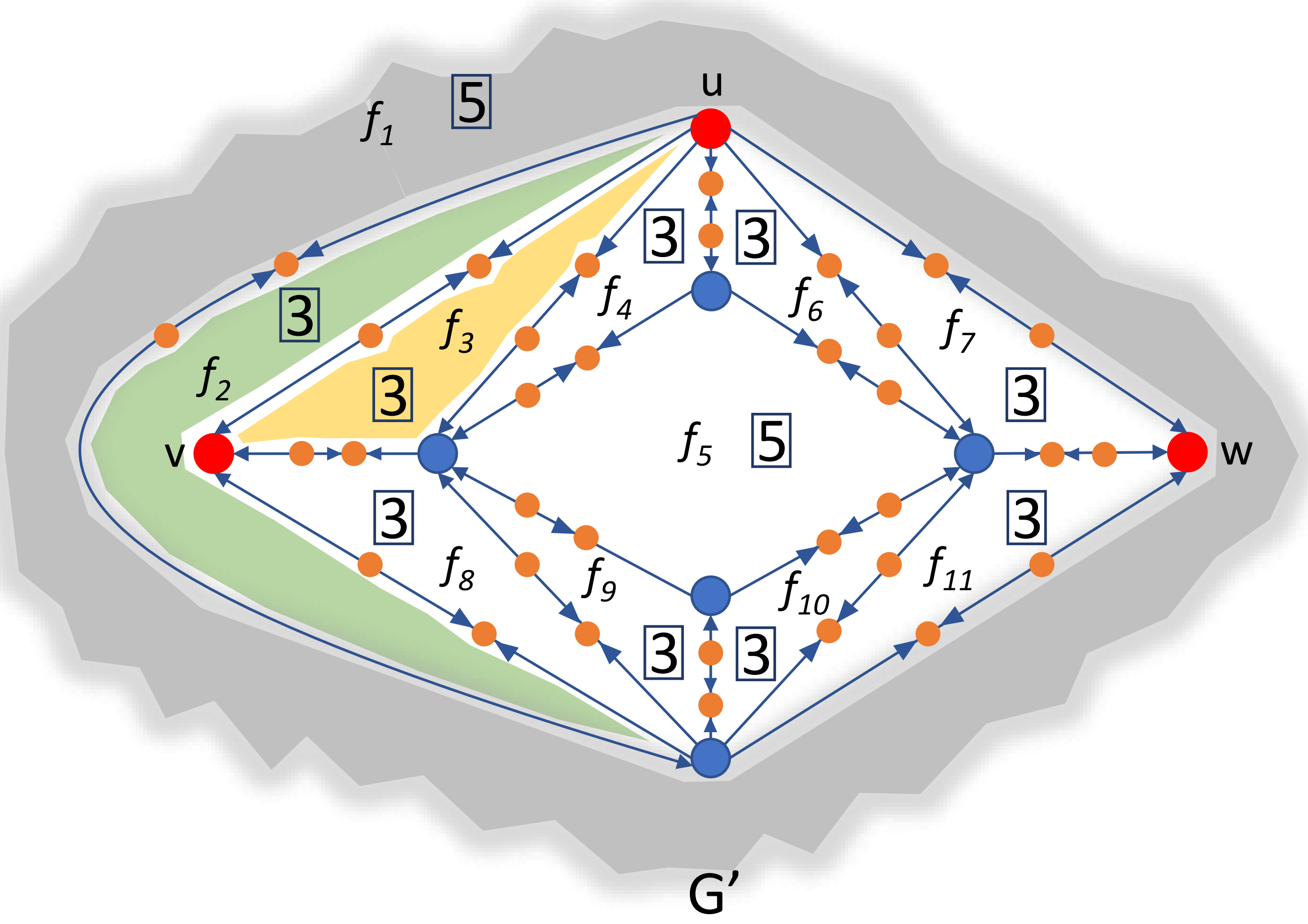
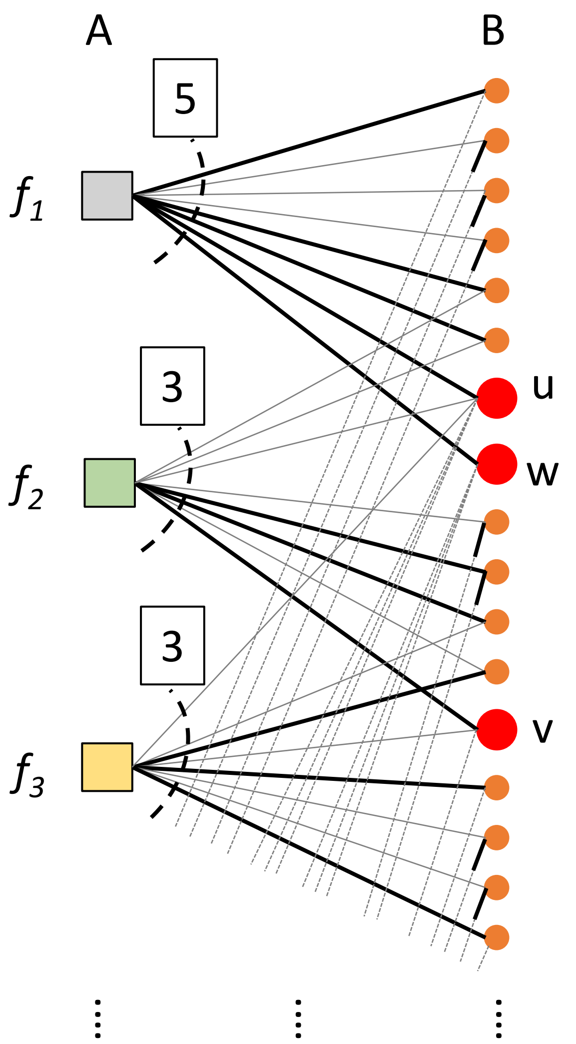
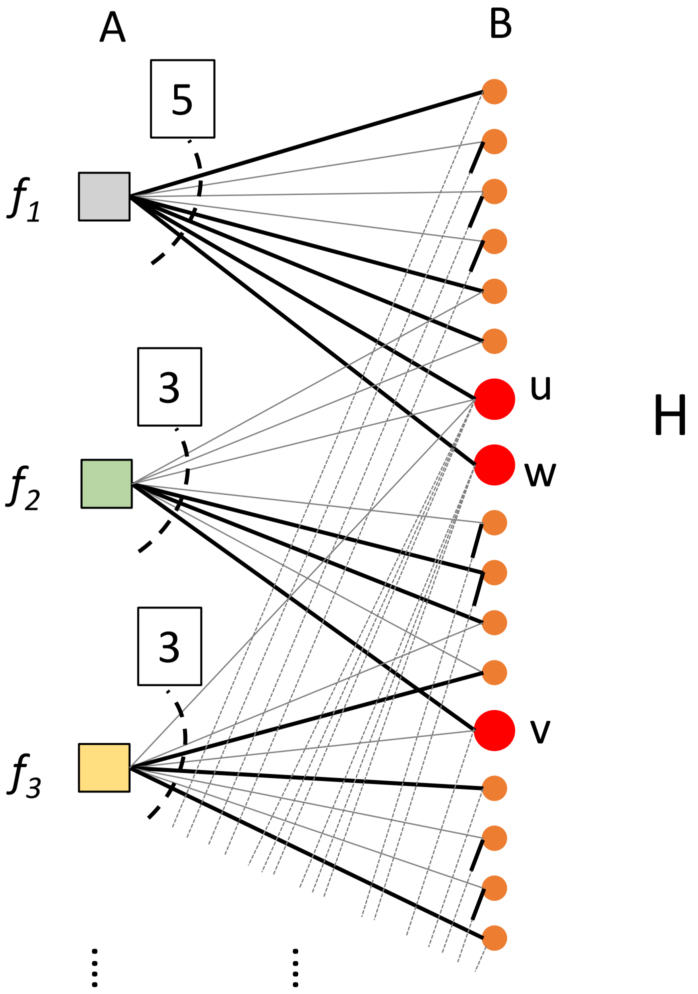


H(A,B,E)



A matching problem

We model our problem as a matching problem.



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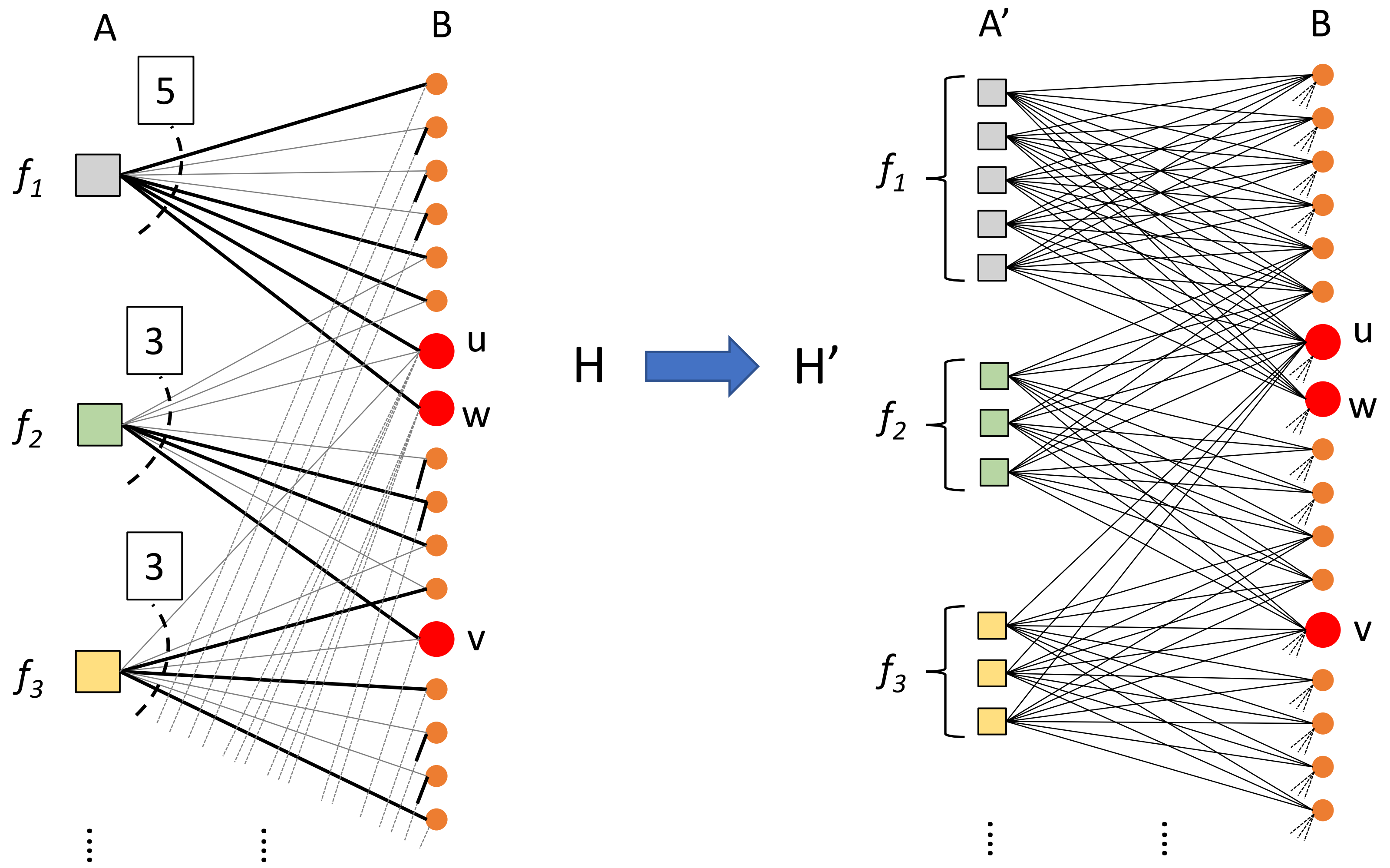
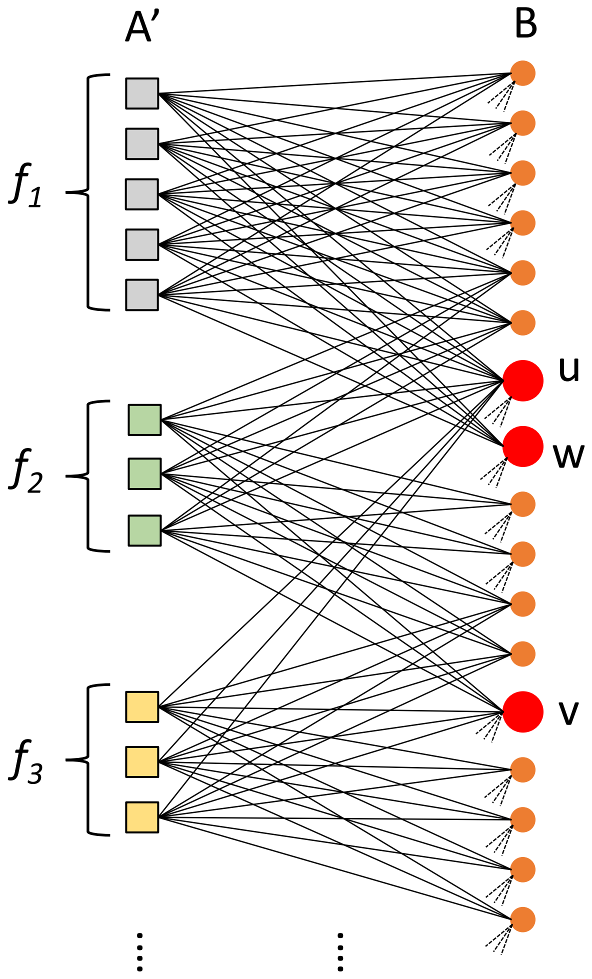


Hall's theorem

We show that H' has a perfect matching by applying the Hall's theorem



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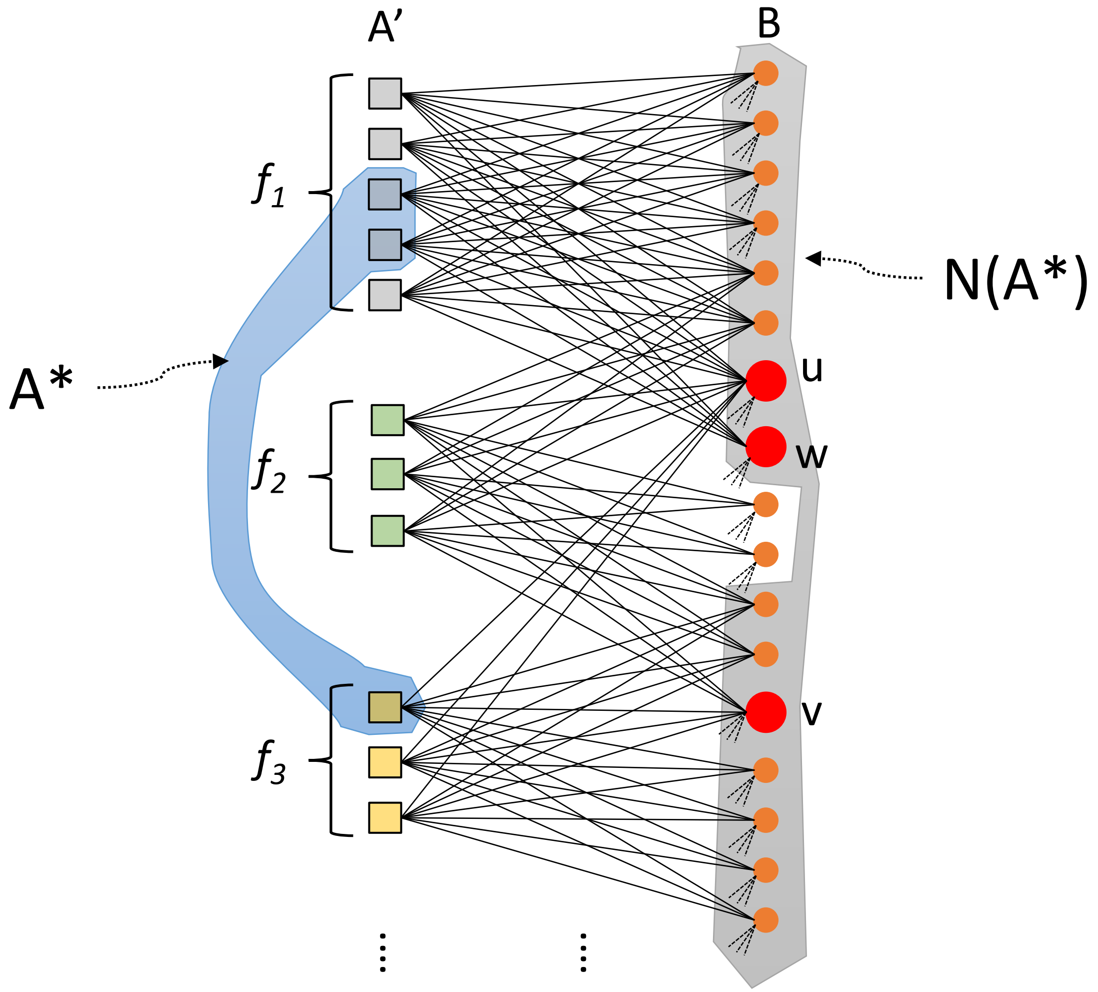
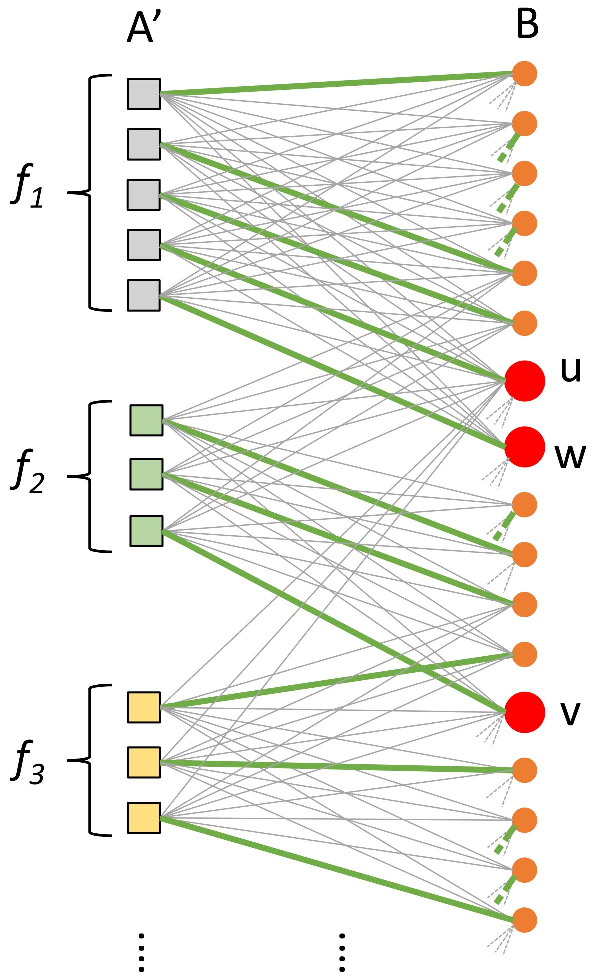
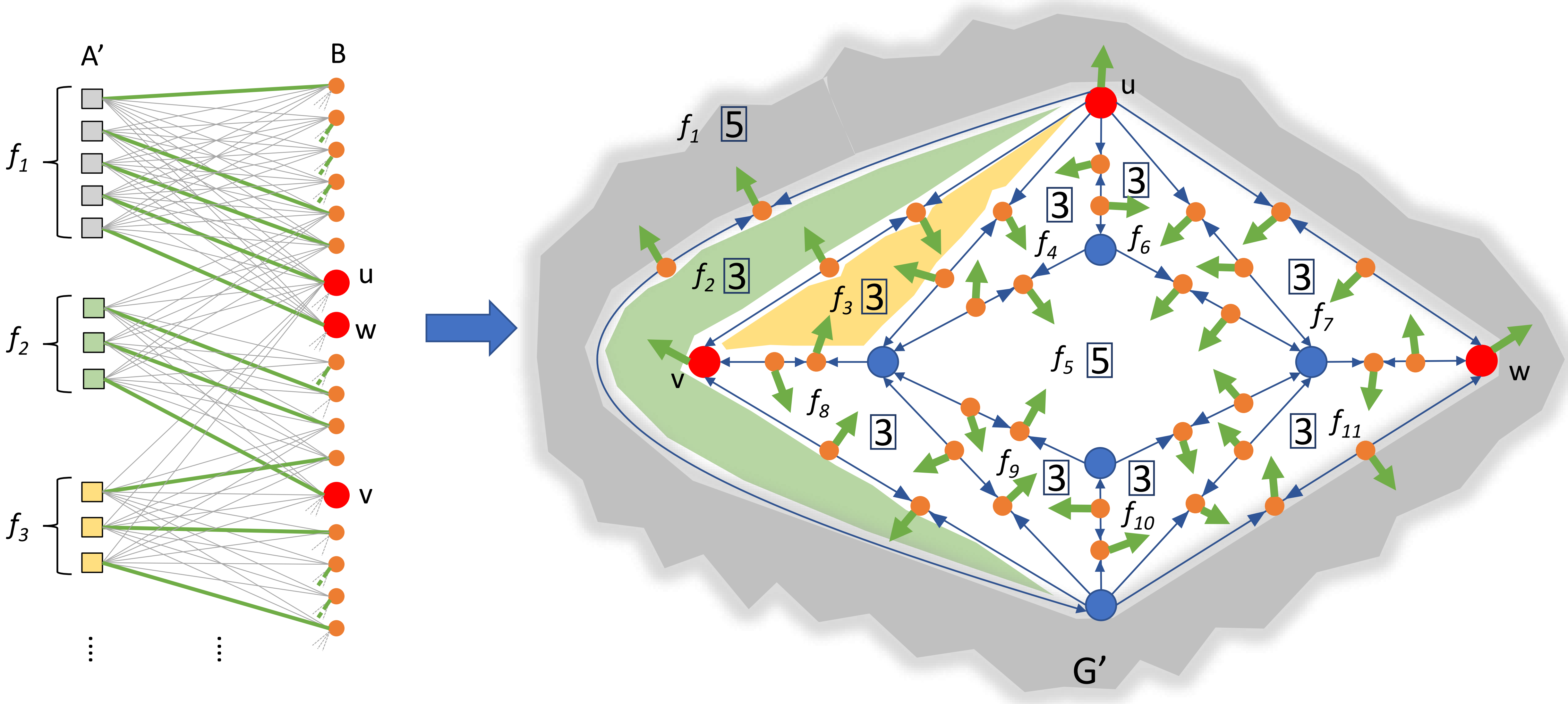


H' has a perfect matching if and only if |A*| ≤ |N(A*)|, for each A* ⊆ A'.

A perfect matching



An upward consistent assignment



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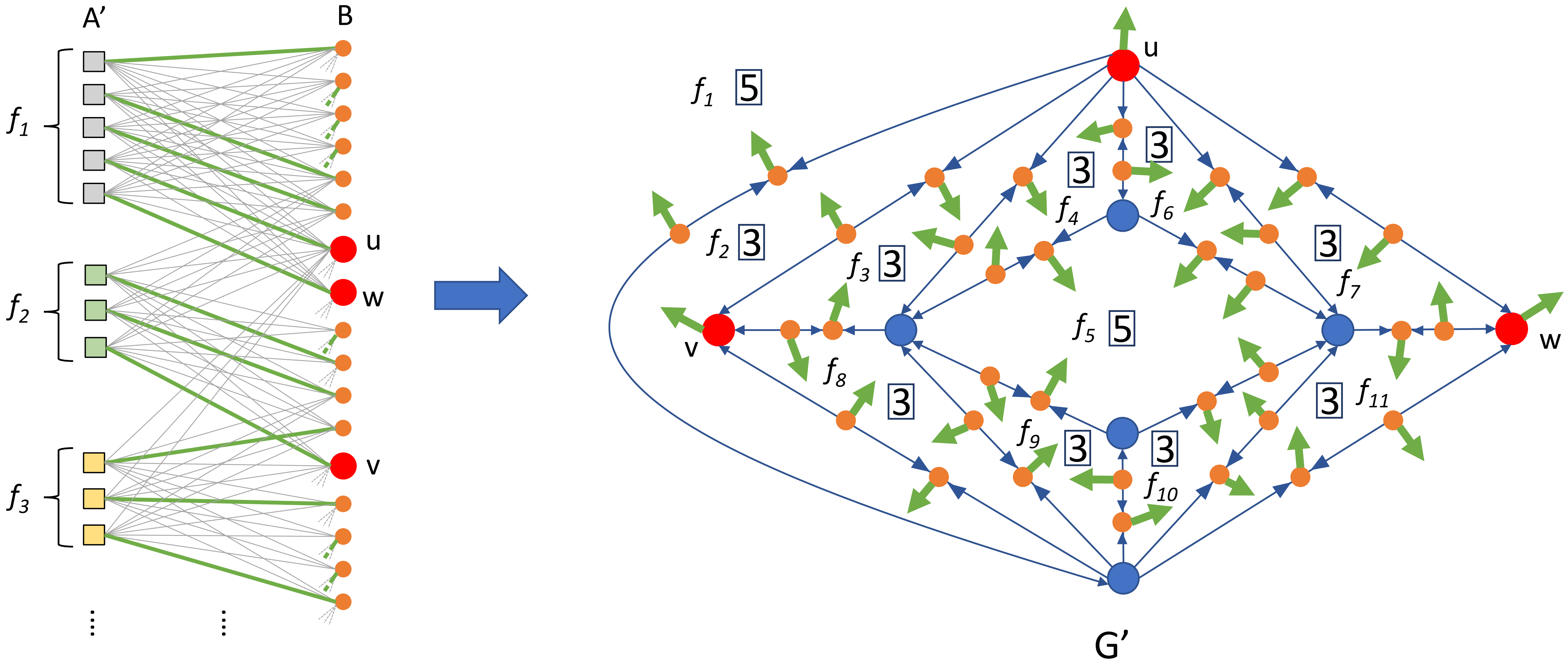
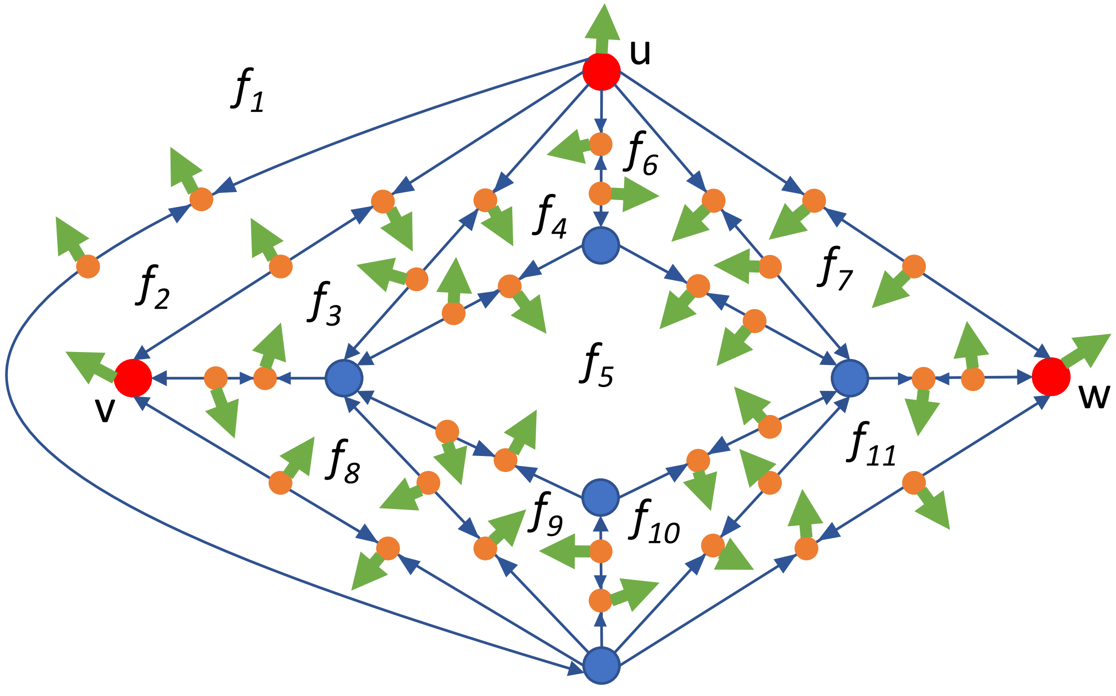


An upward planar drawing of G’



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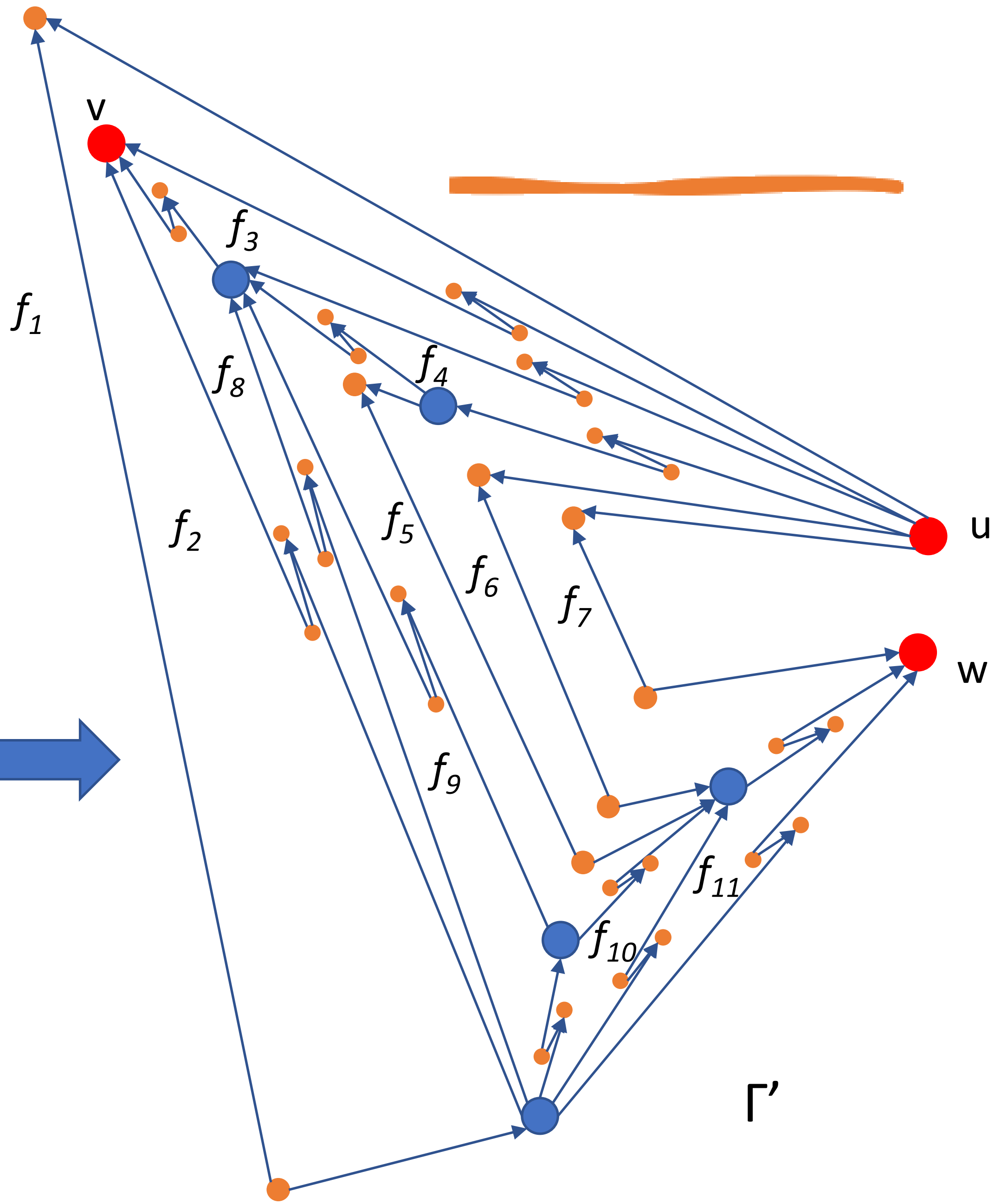
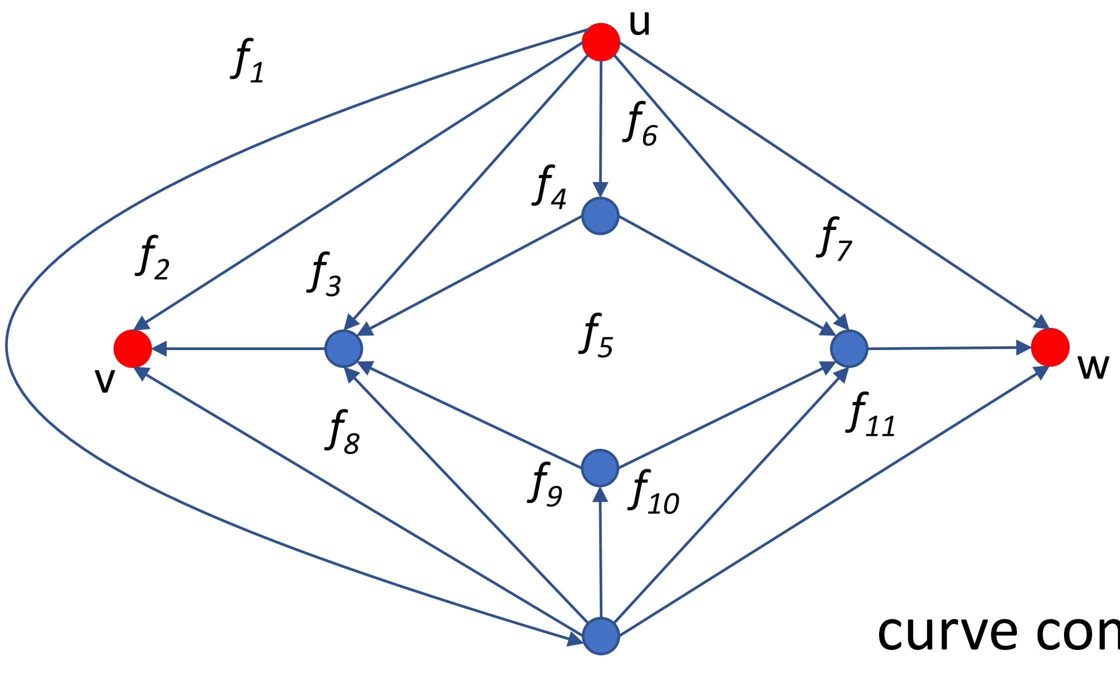


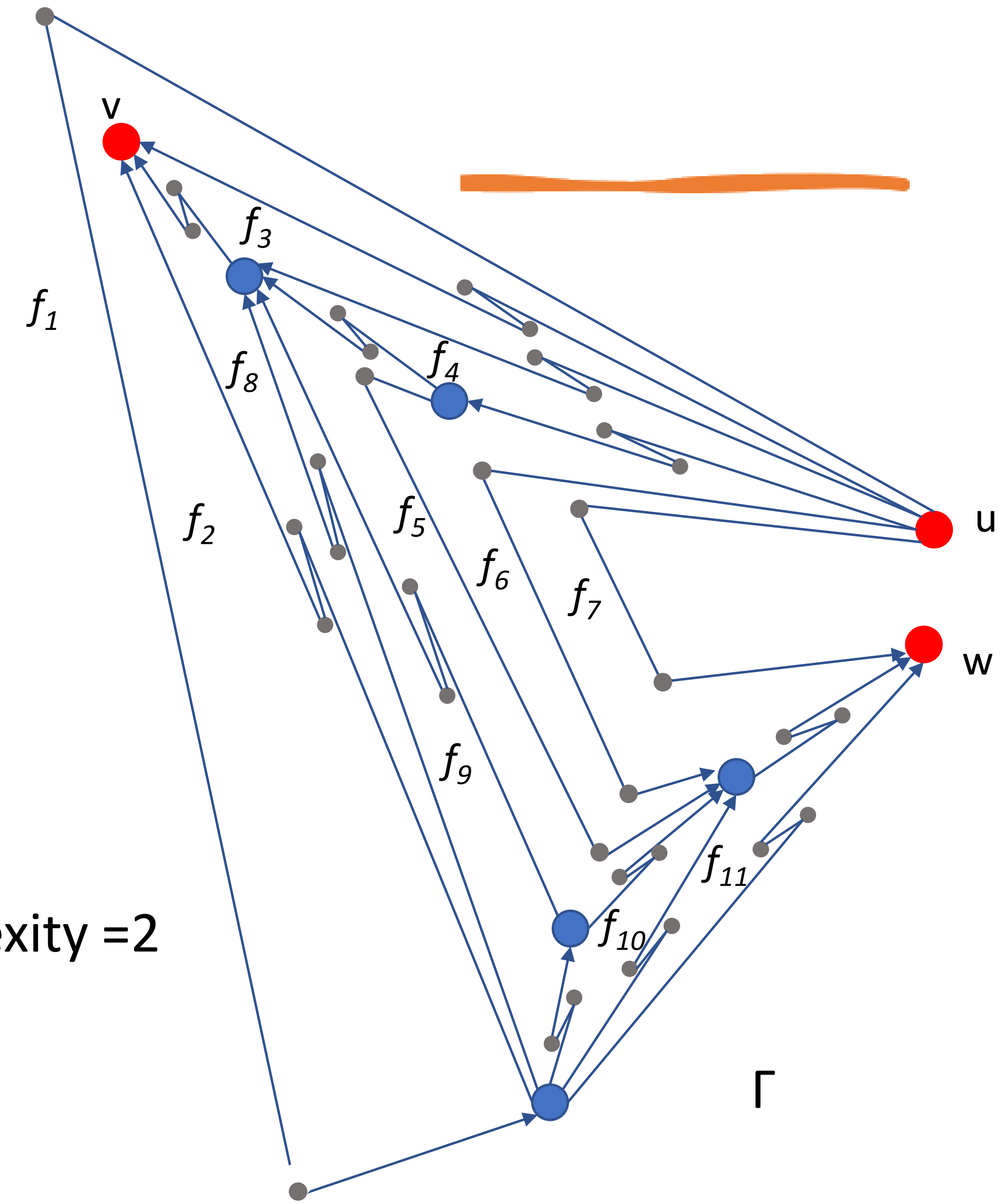
Γ’

A quasi-upward planar drawing of G



G

curve complexity =2



Γ

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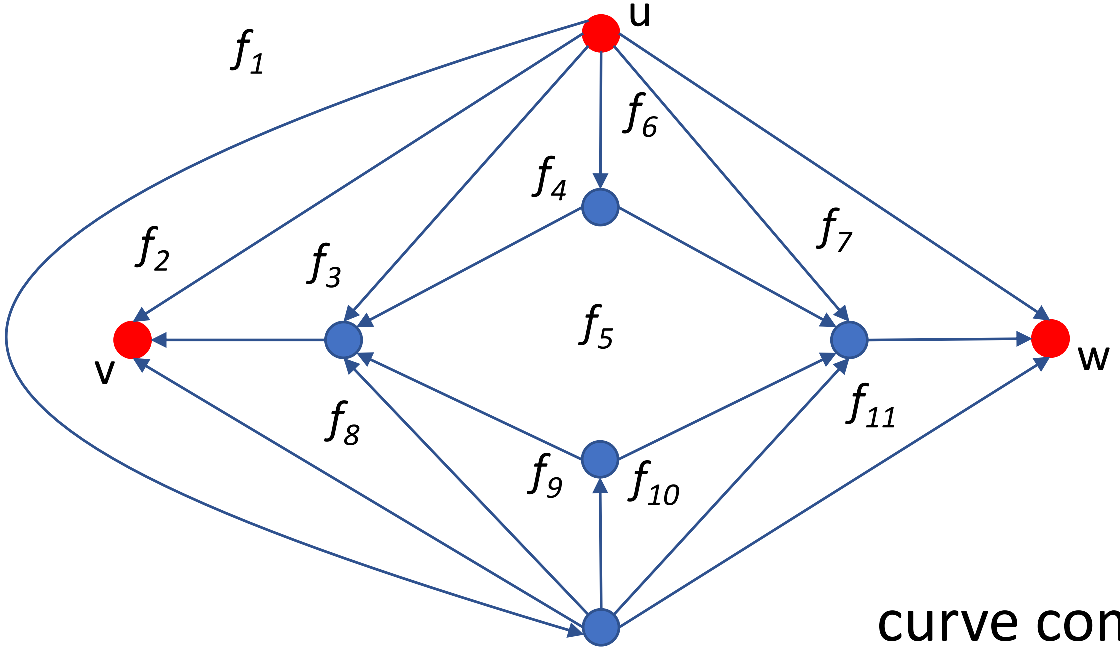

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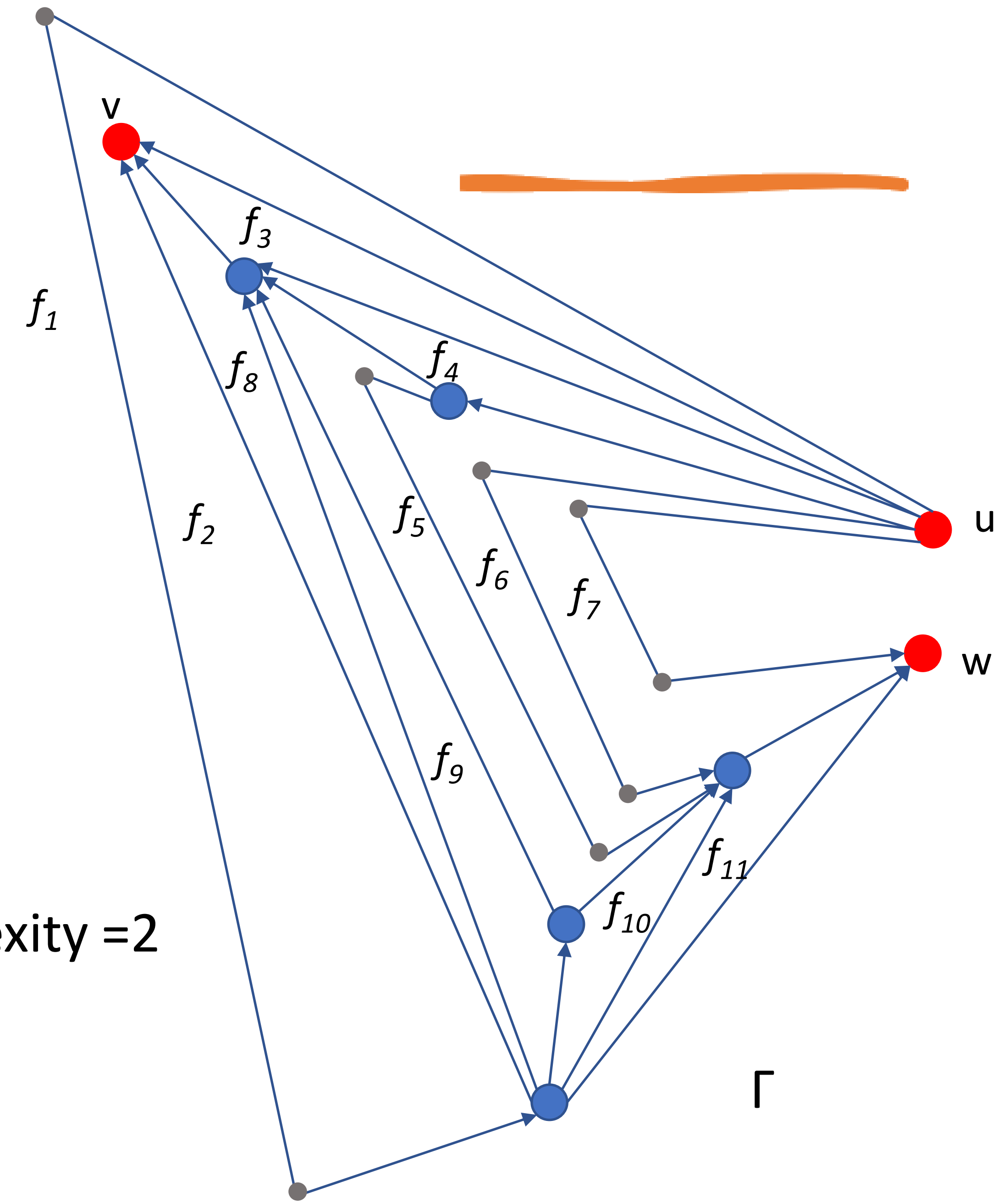
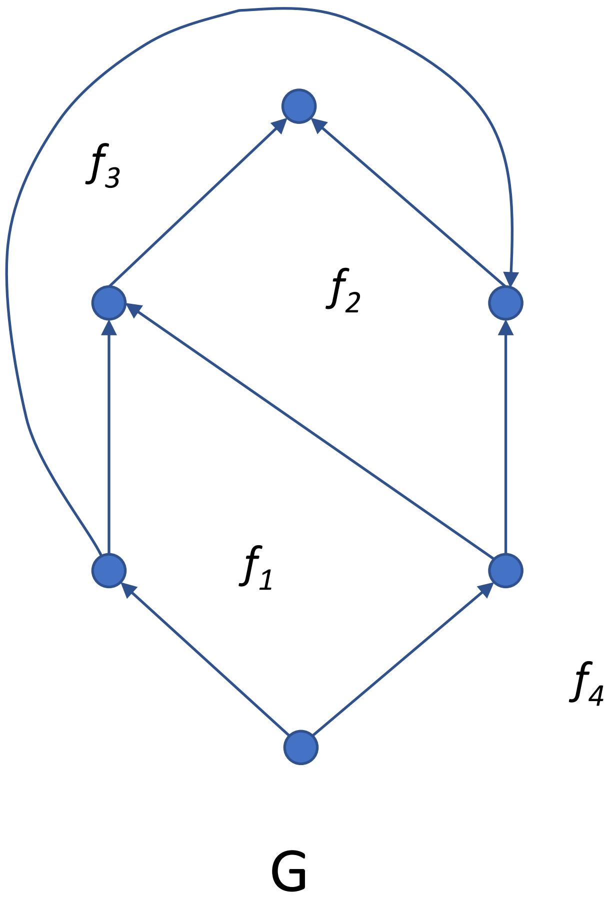


Γ

Computing Minimum Curve Complexity Drawings

A unit-capacity flow network

We define a variant of the flow network used by Bertolazzi et al. [Bertolazzi, Di Battista, Didimo - Quasi-Upward planarity. Algorithmica. 2002]



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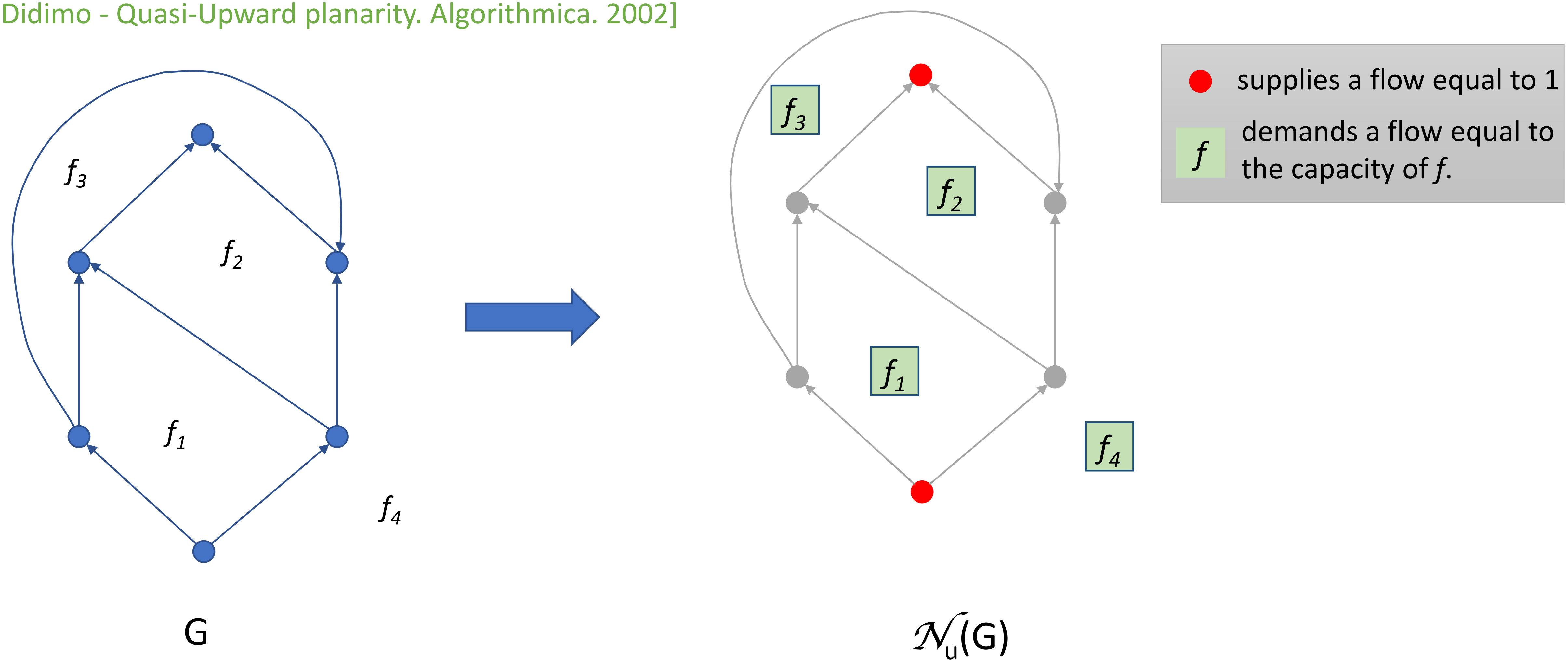

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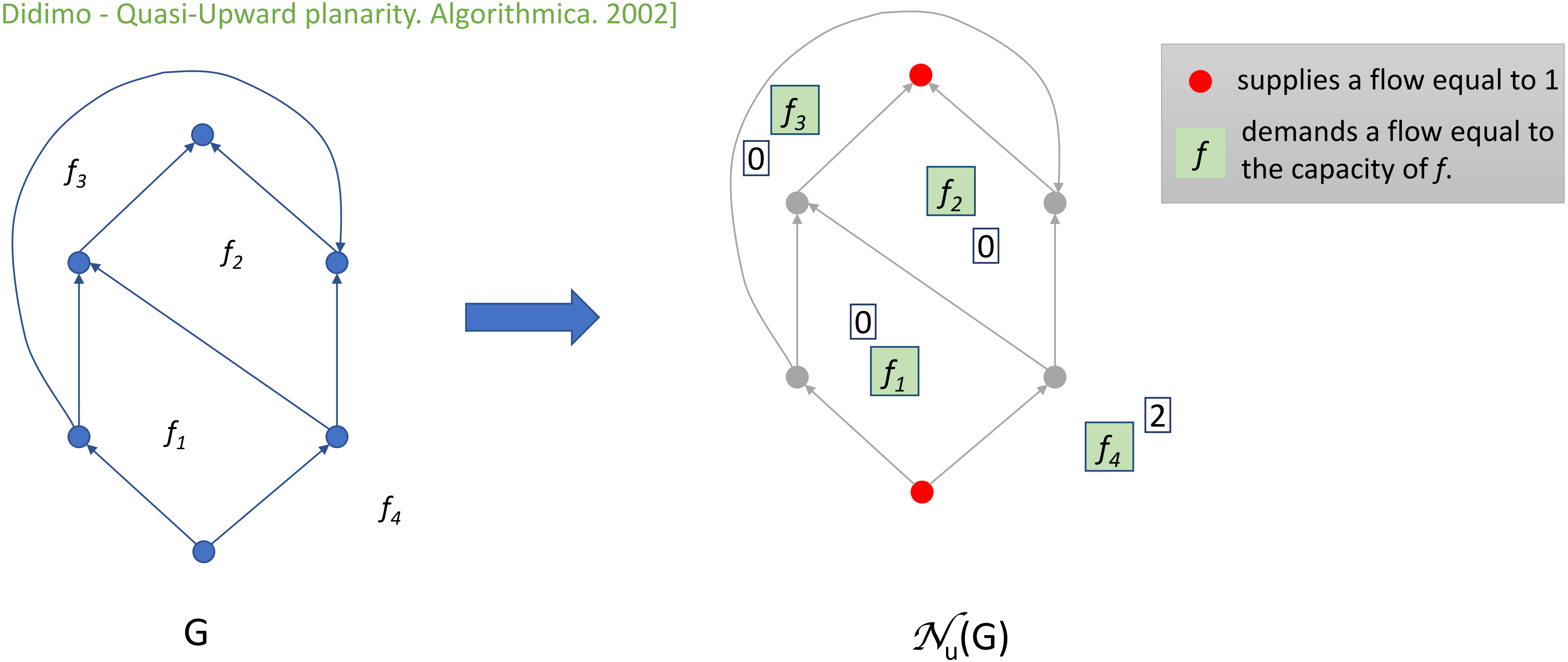

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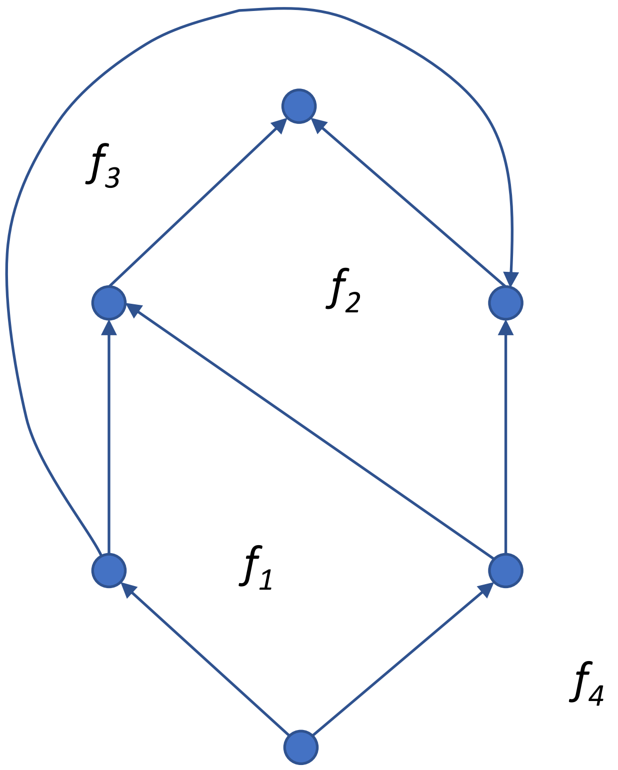
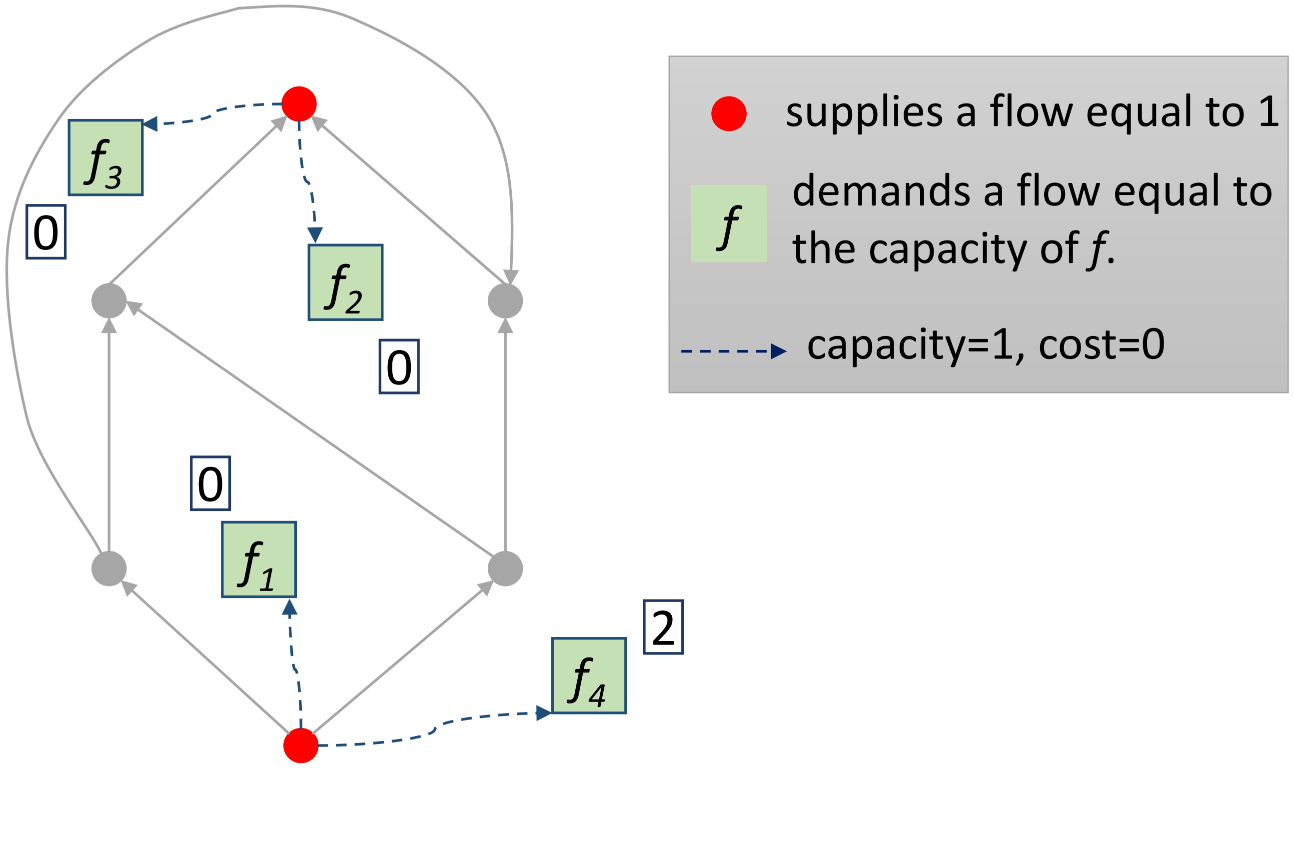


G



$\mathcal{N}_u$(G)

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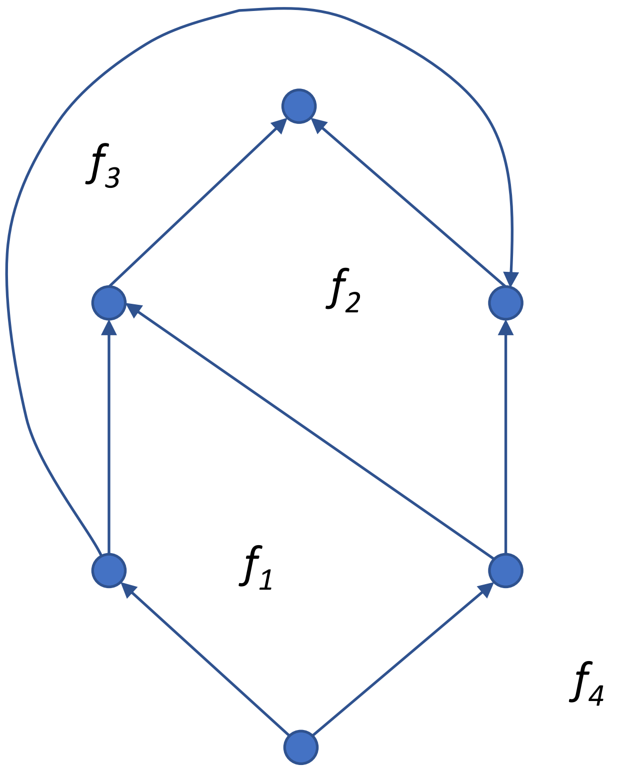
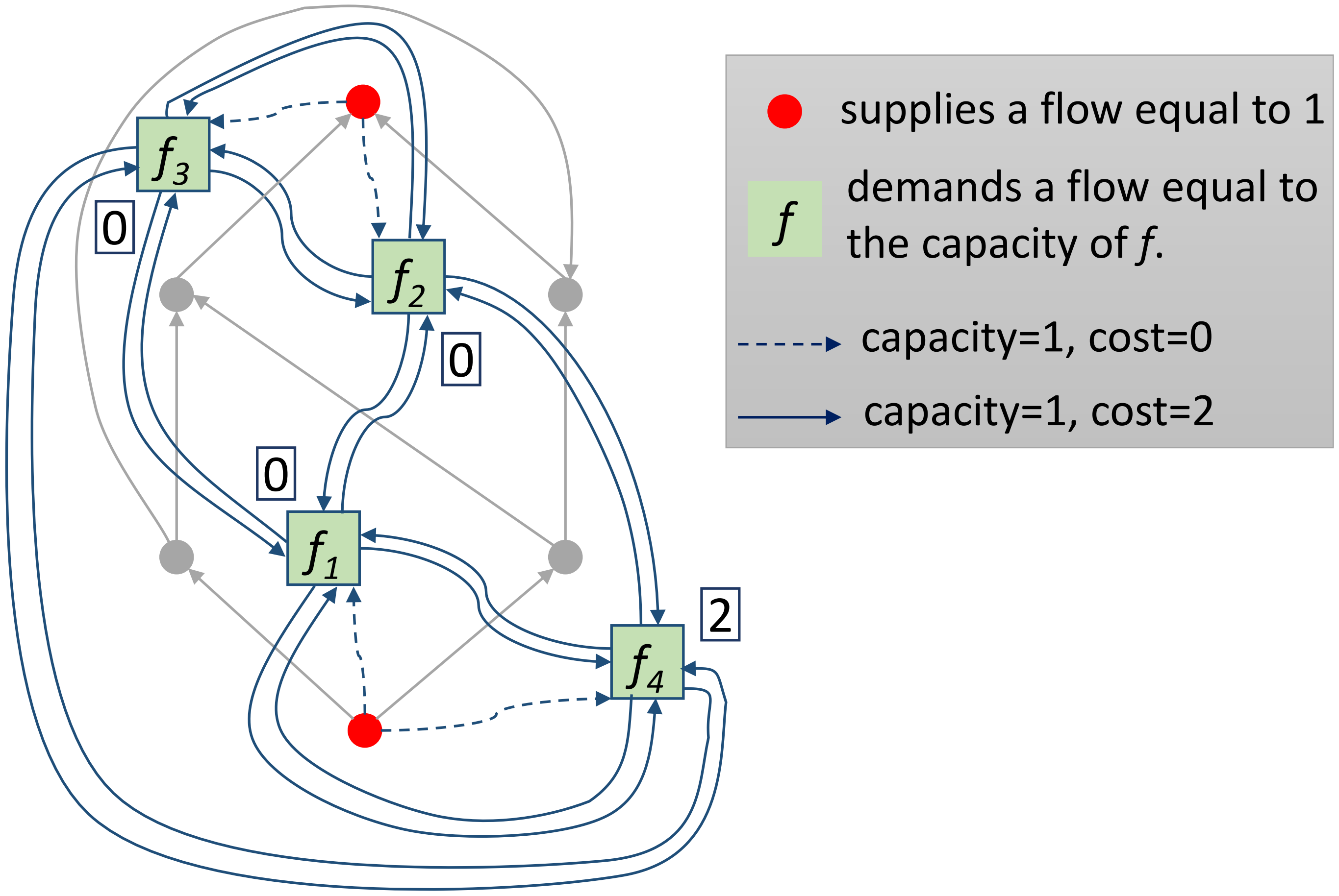
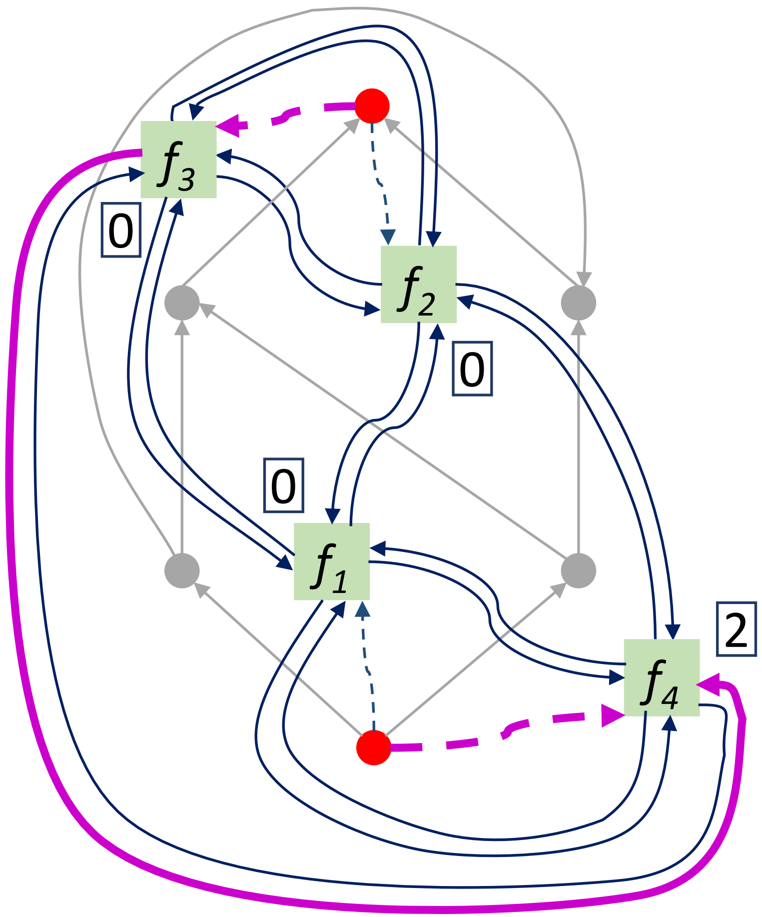


G

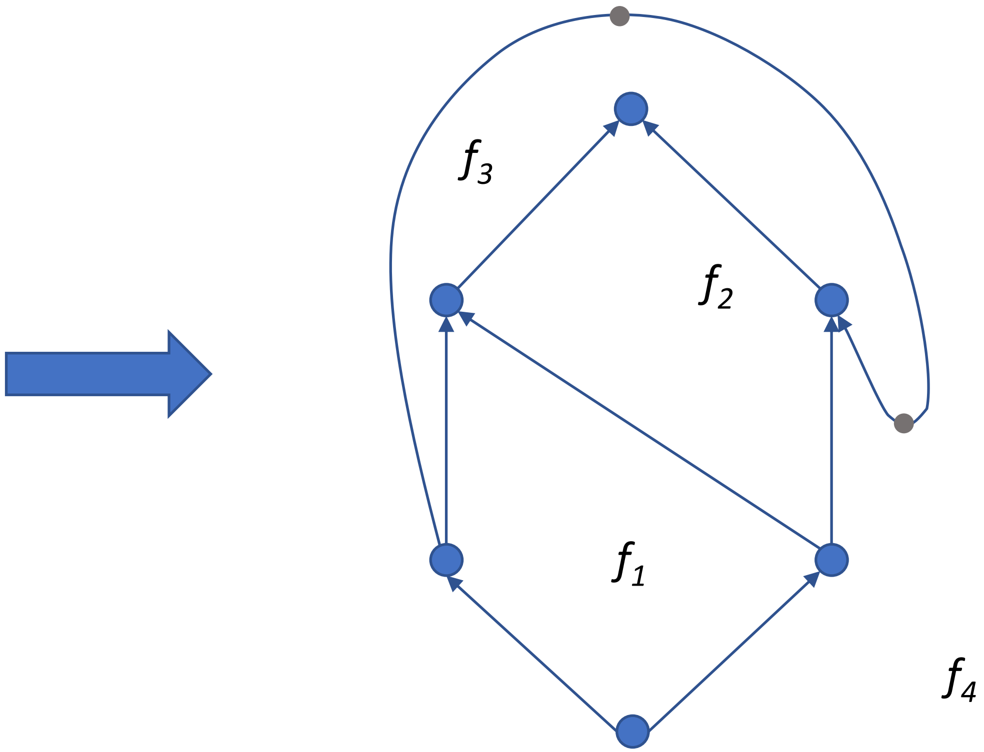


$\mathcal{N}_u$(G)

A unit-capacity flow network



$\mathcal{N}_u$(G)



Γ

A unit-capacity flow network

We use the algorithm by Karczmarz and Sankowski whose time complexity is $\tilde{O}((NM)^{\frac{2}{3}} \log C)$.

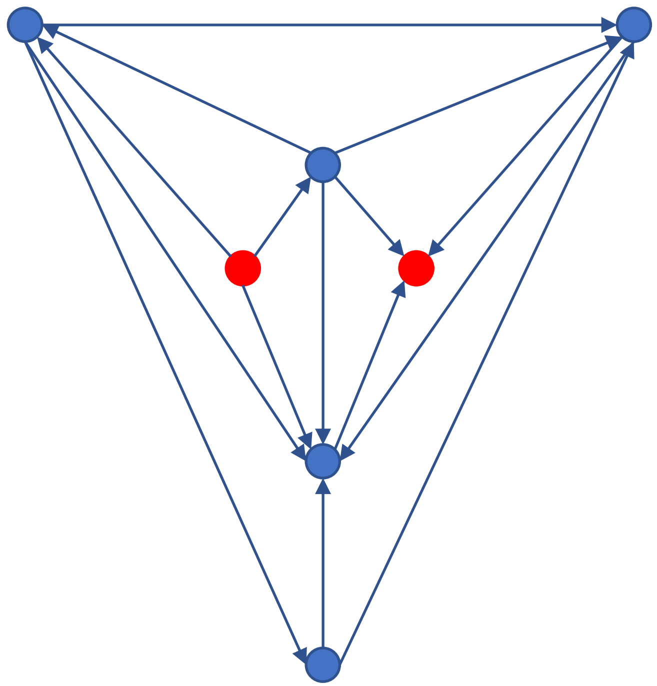
N and M are the number of vertices and edges of the flow network, respectively.
C is an upper bound to the edge costs.
[Karczmarz, Sankowski: Min-cost flow in unit-capacity planar graphs. ESA 2019]

For $\mathcal{N}_u$(G) we have $N \in O(m)$, $M \in O(m)$ and $C \in O(1)$.

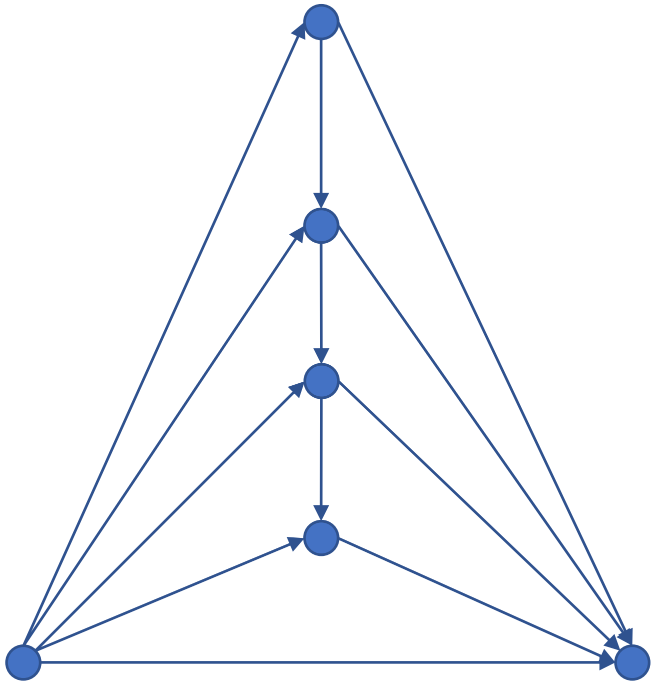
Our algorithm has time complexity $\tilde{O}(m^{\frac{4}{3}})$.

A Lower Bound on the Curve Complexity

Gadgets



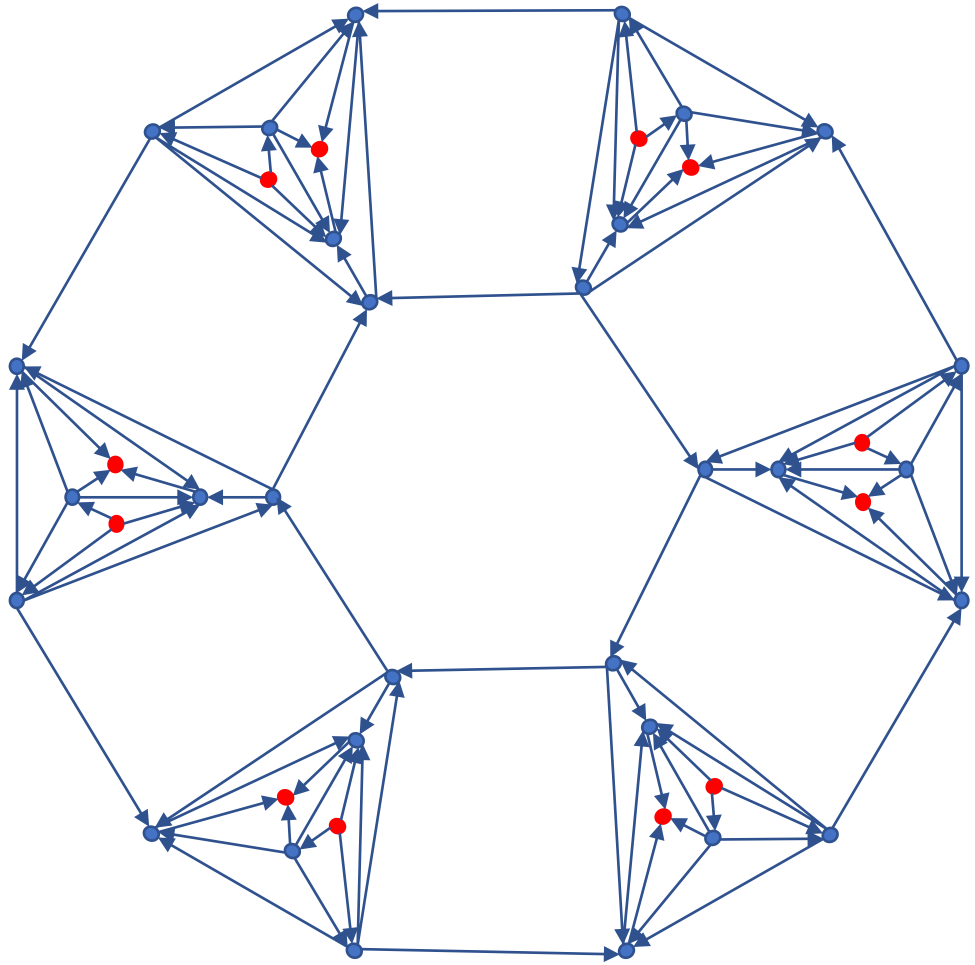
Supplier gadget



Barrier gadget

G_6

k supplier gadgets



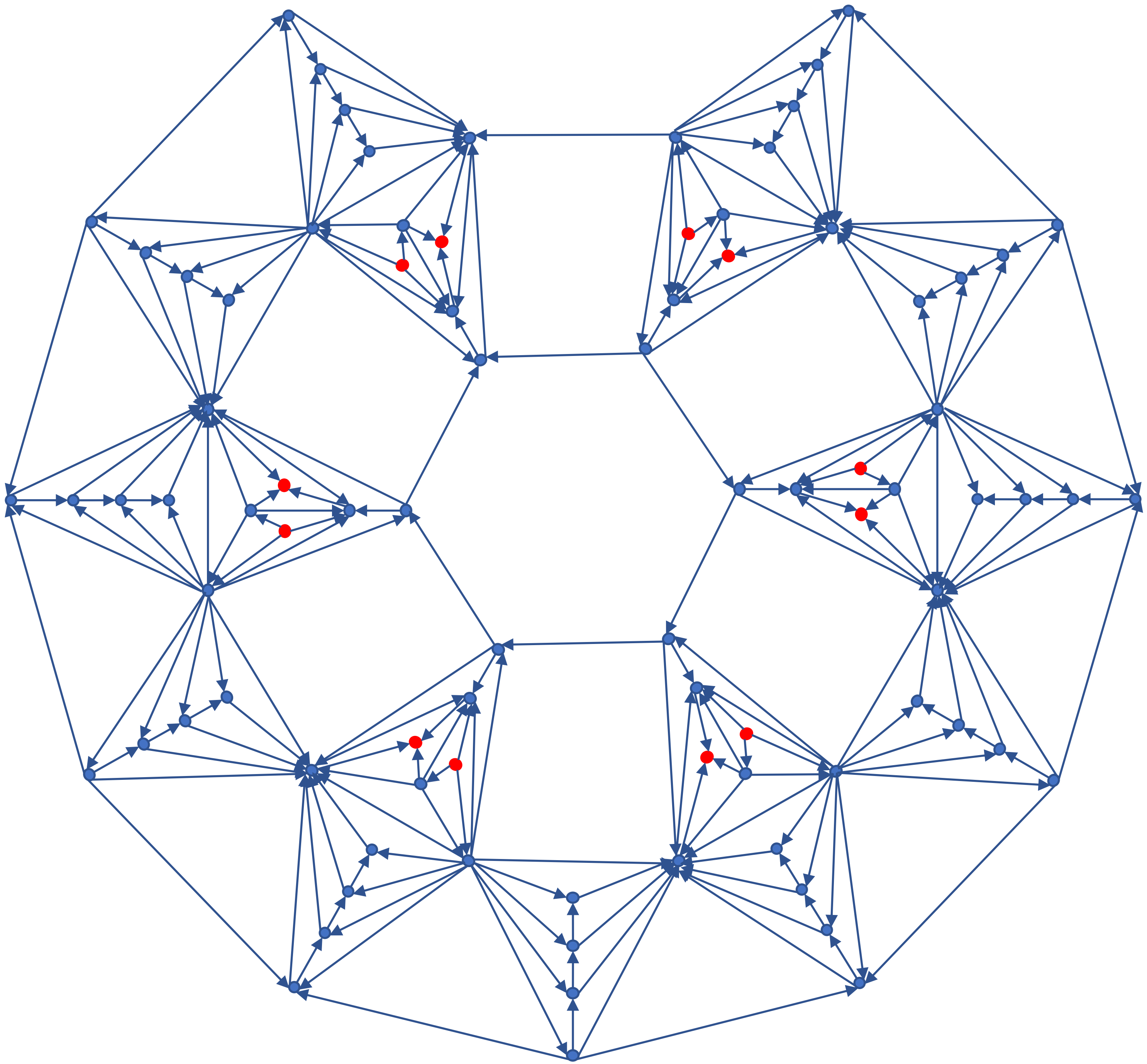
G_6

k supplier gadgets

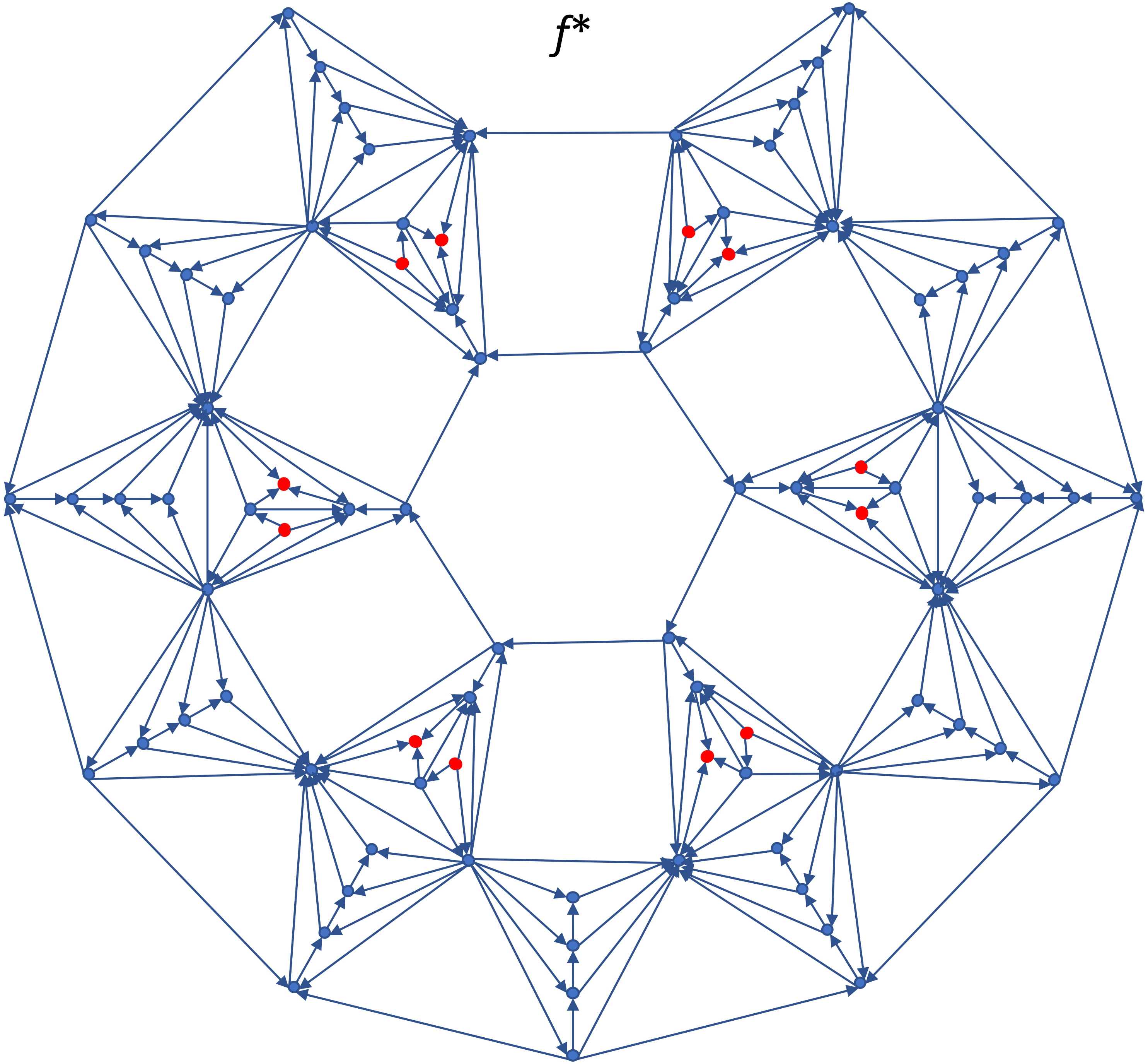
+

2k-1 barrier gadgets

G_k is acyclic
and
triconnected

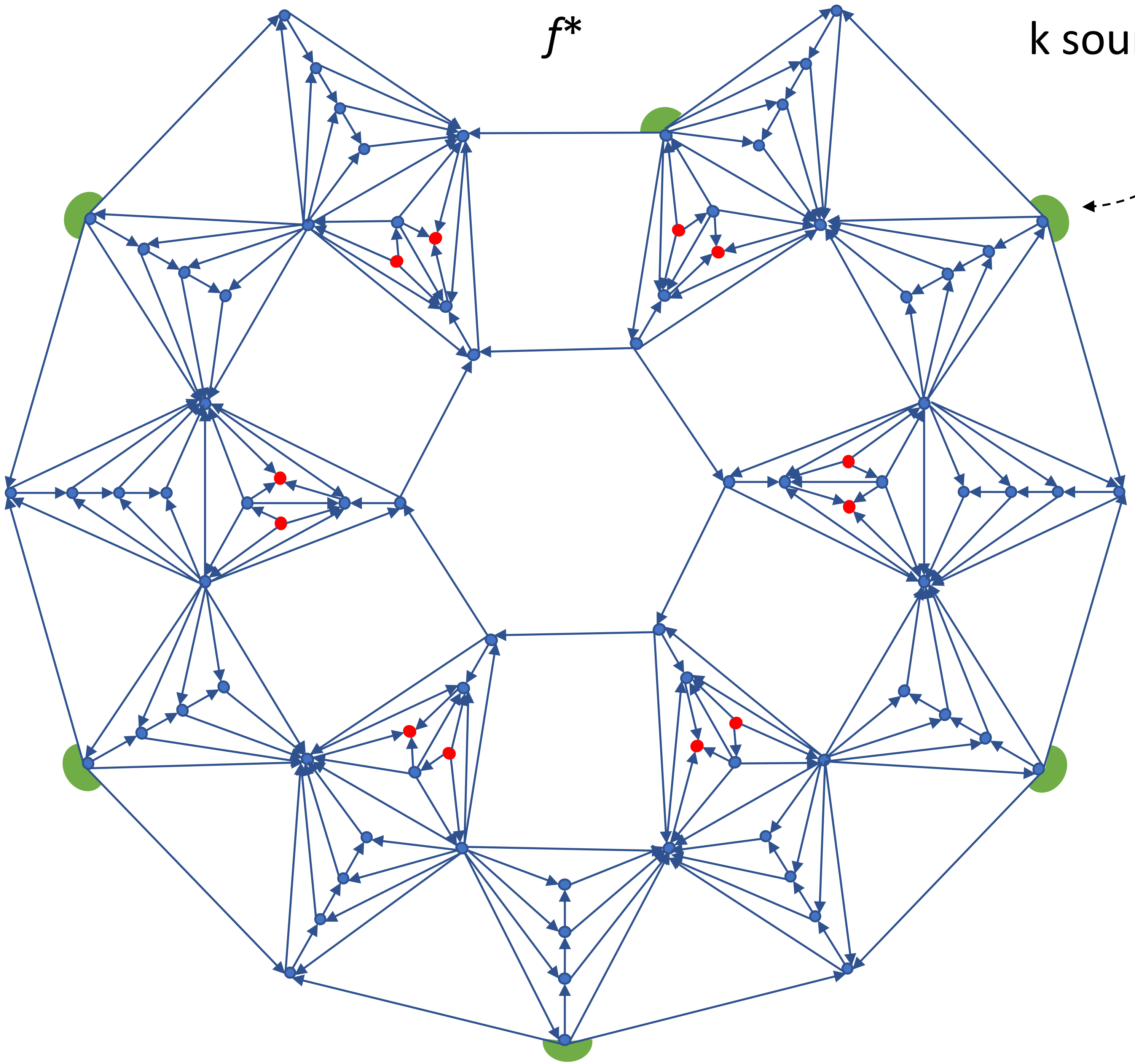


G_6



G_6

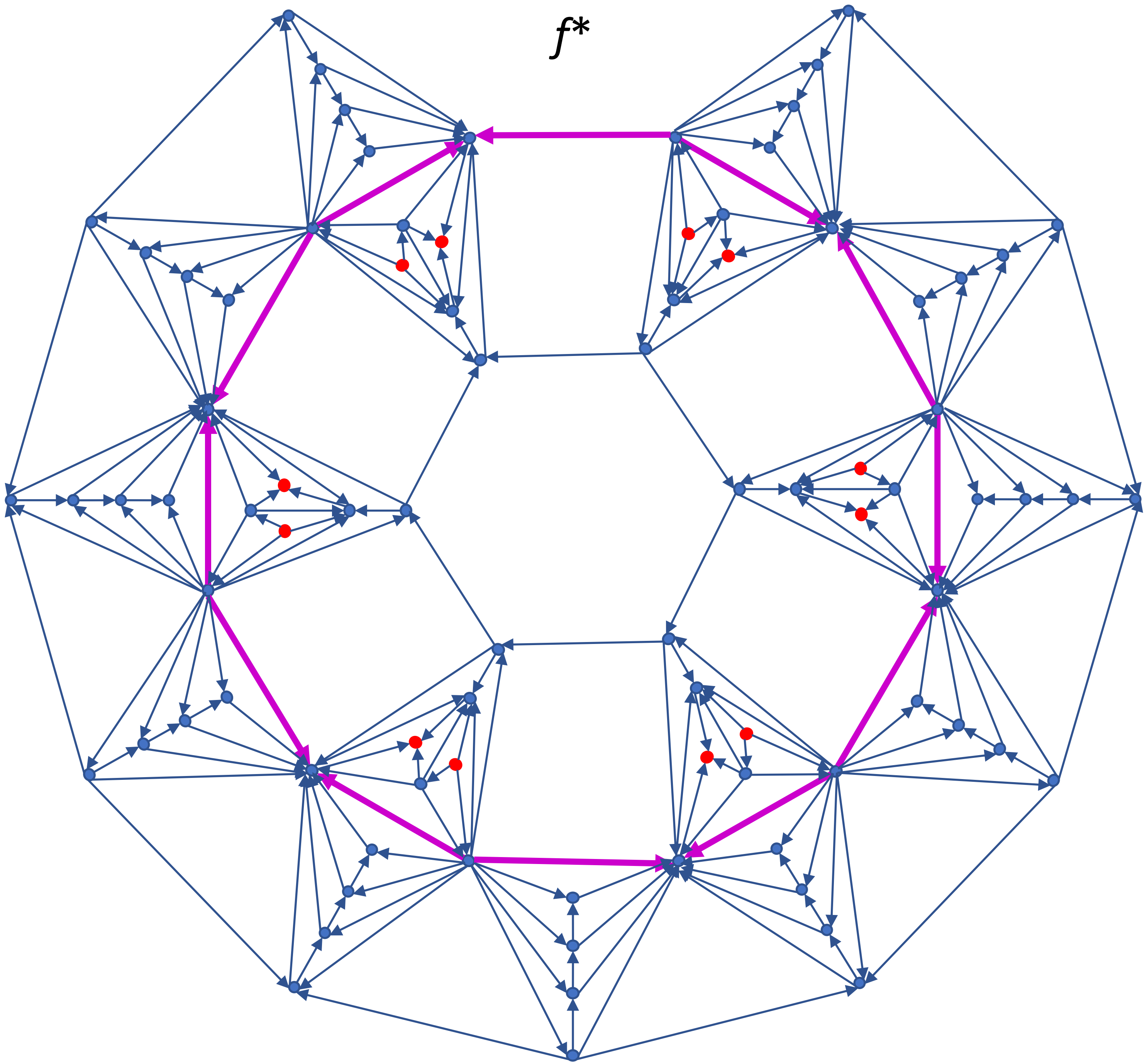
f^* has capacity at least k-1.



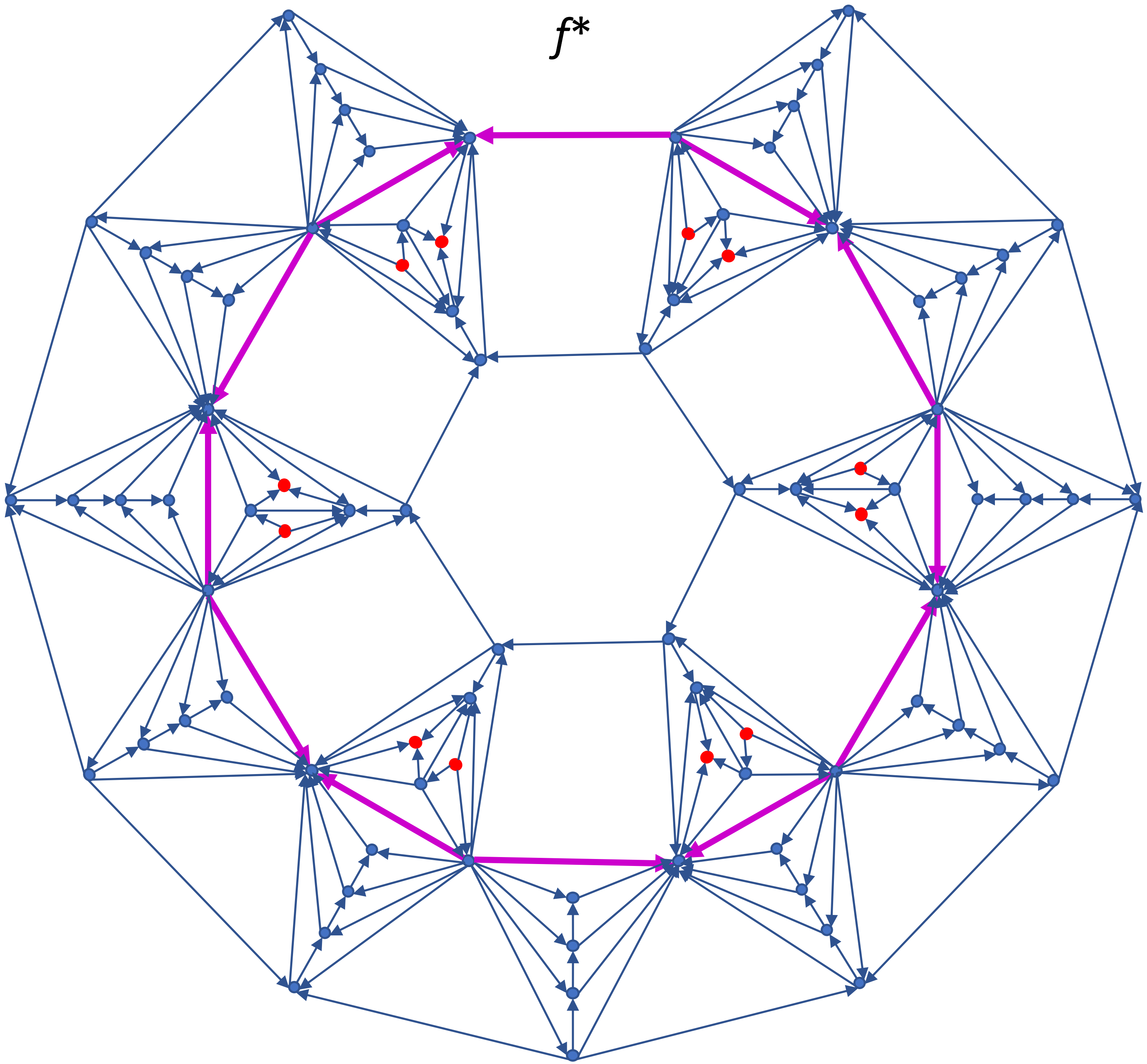
k source-switches

G_6

f^* has capacity at least k-1.



G_6



f^* has capacity at least k-1.

↓

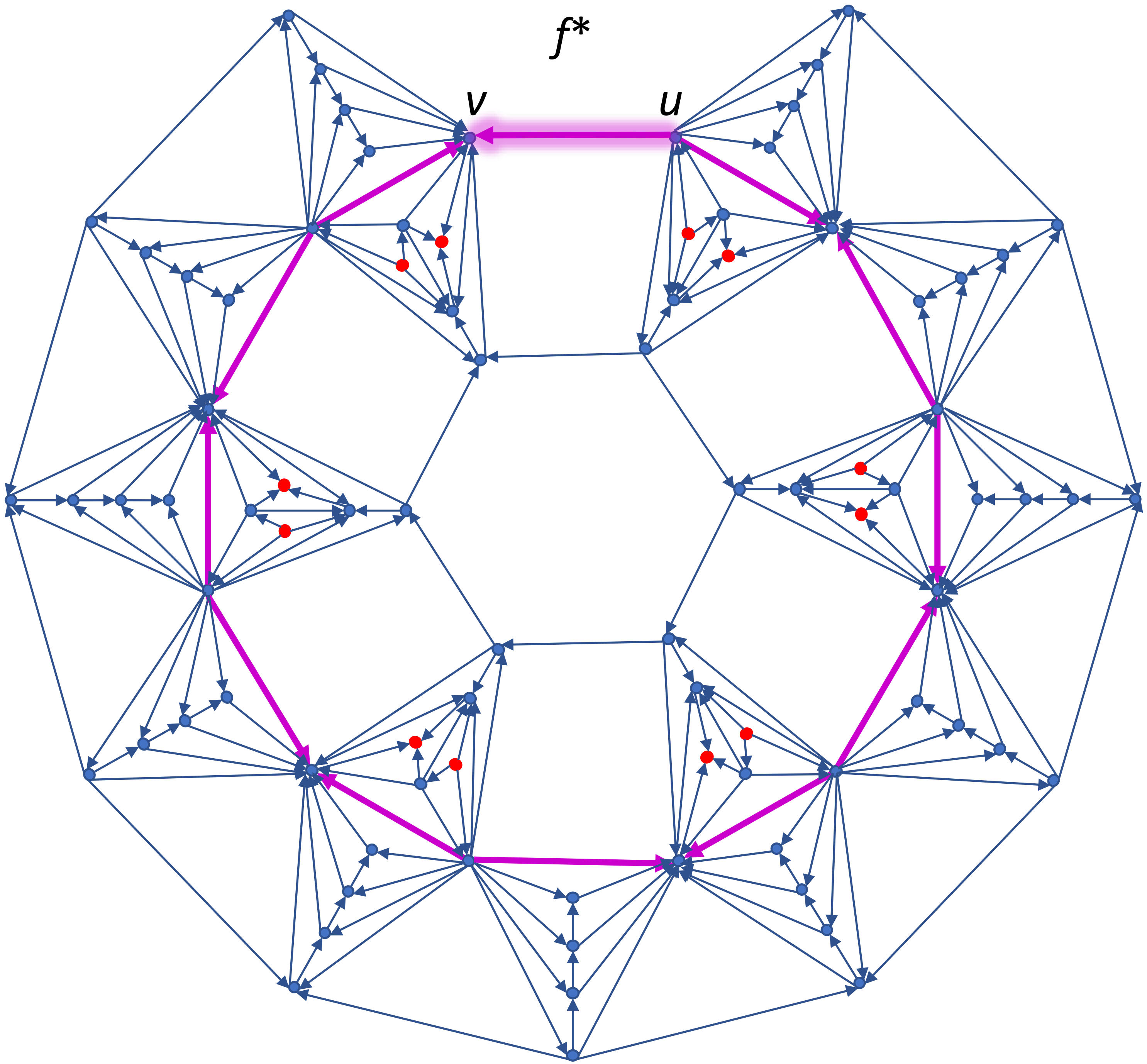
At least k – 1 units of flow must traverse the purple edges.

G_6

f^* has capacity at least k-1.

↓

At least k – 1 units of flow must traverse edge (u,v).

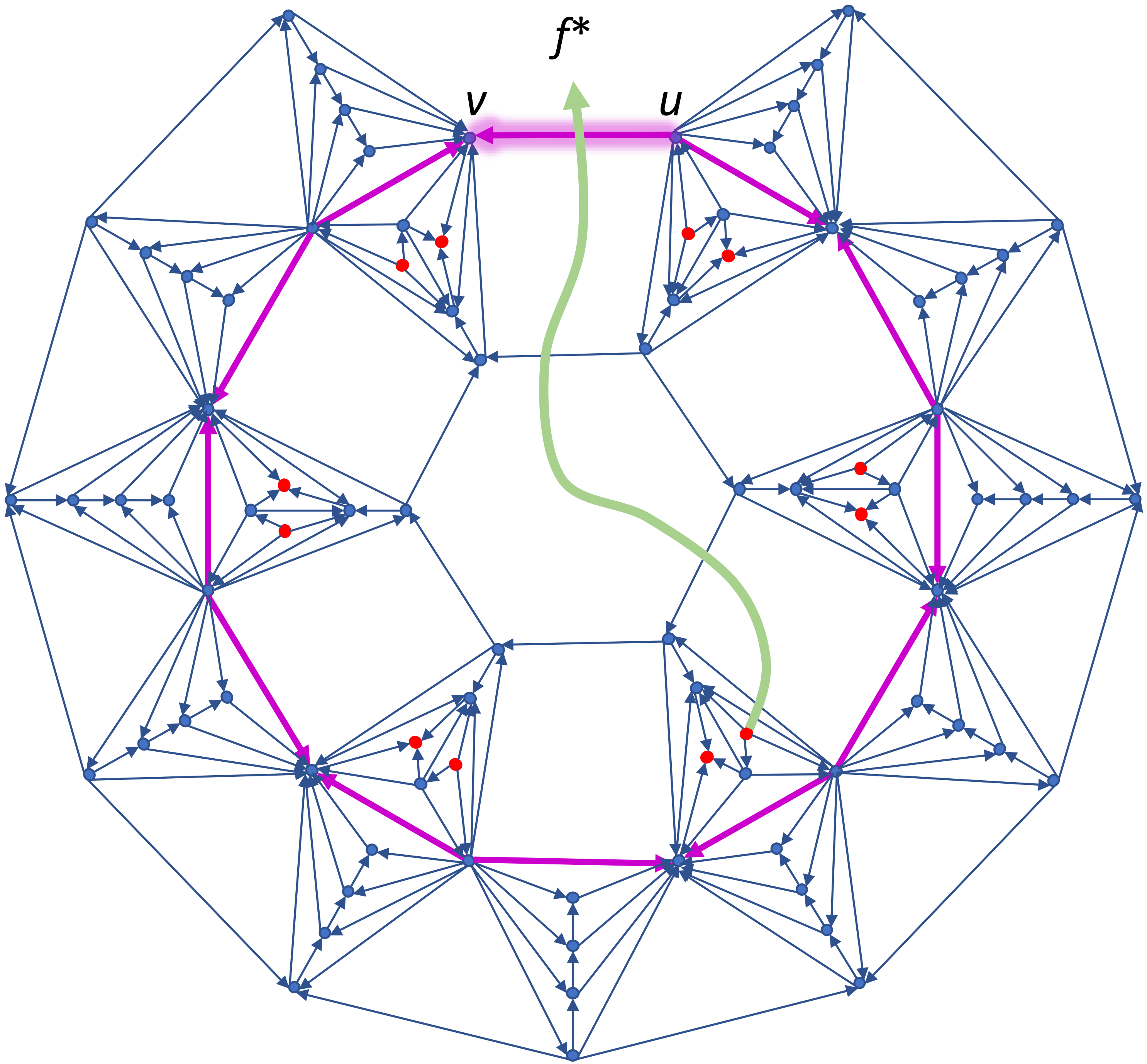


G_6

f^* has capacity at least k-1.

↓

At least k – 1 units of flow must traverse edge (u,v).

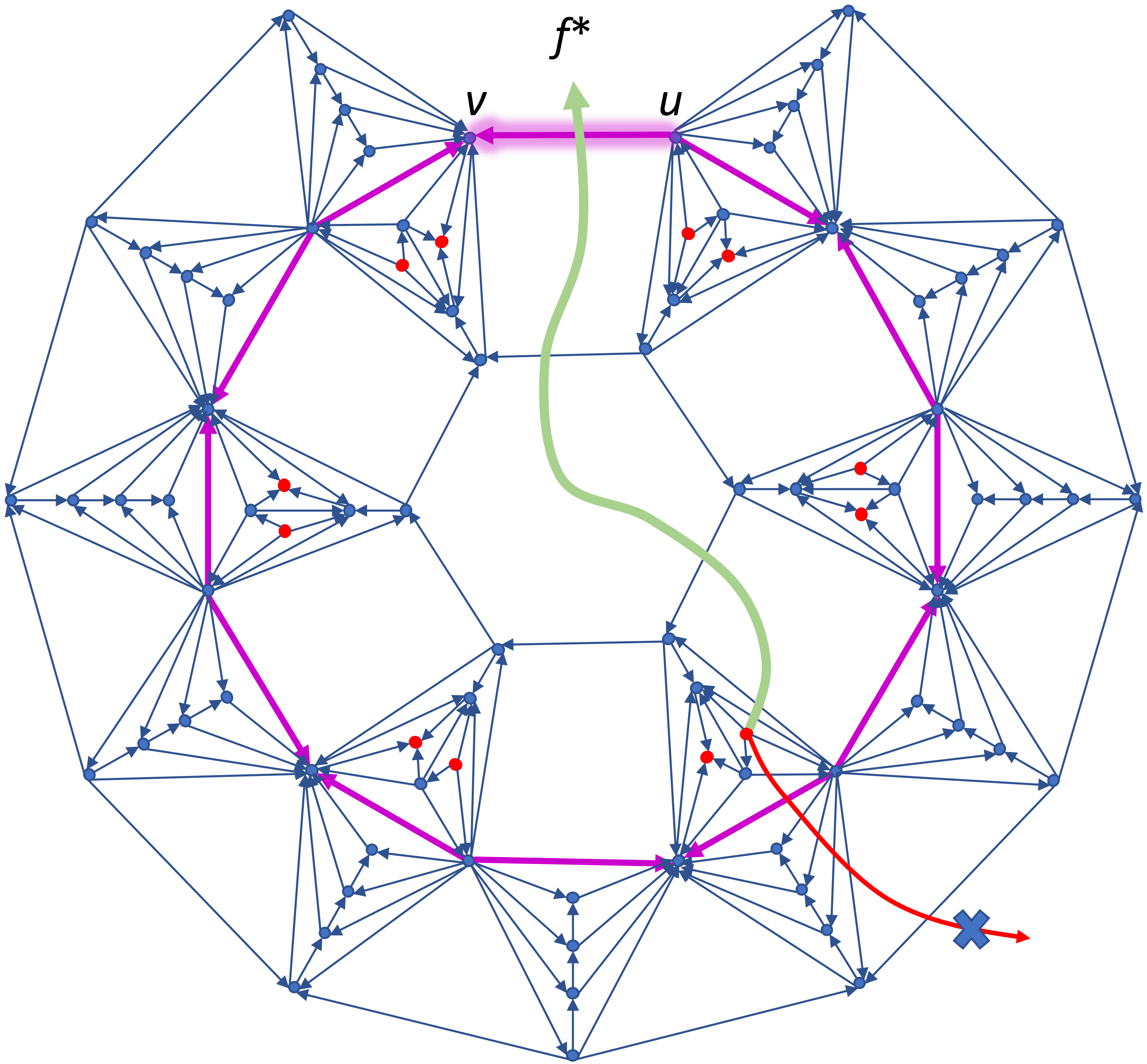


G_6

f^* has capacity at least k-1.

↓

At least k – 1 units of flow must traverse edge (u,v).

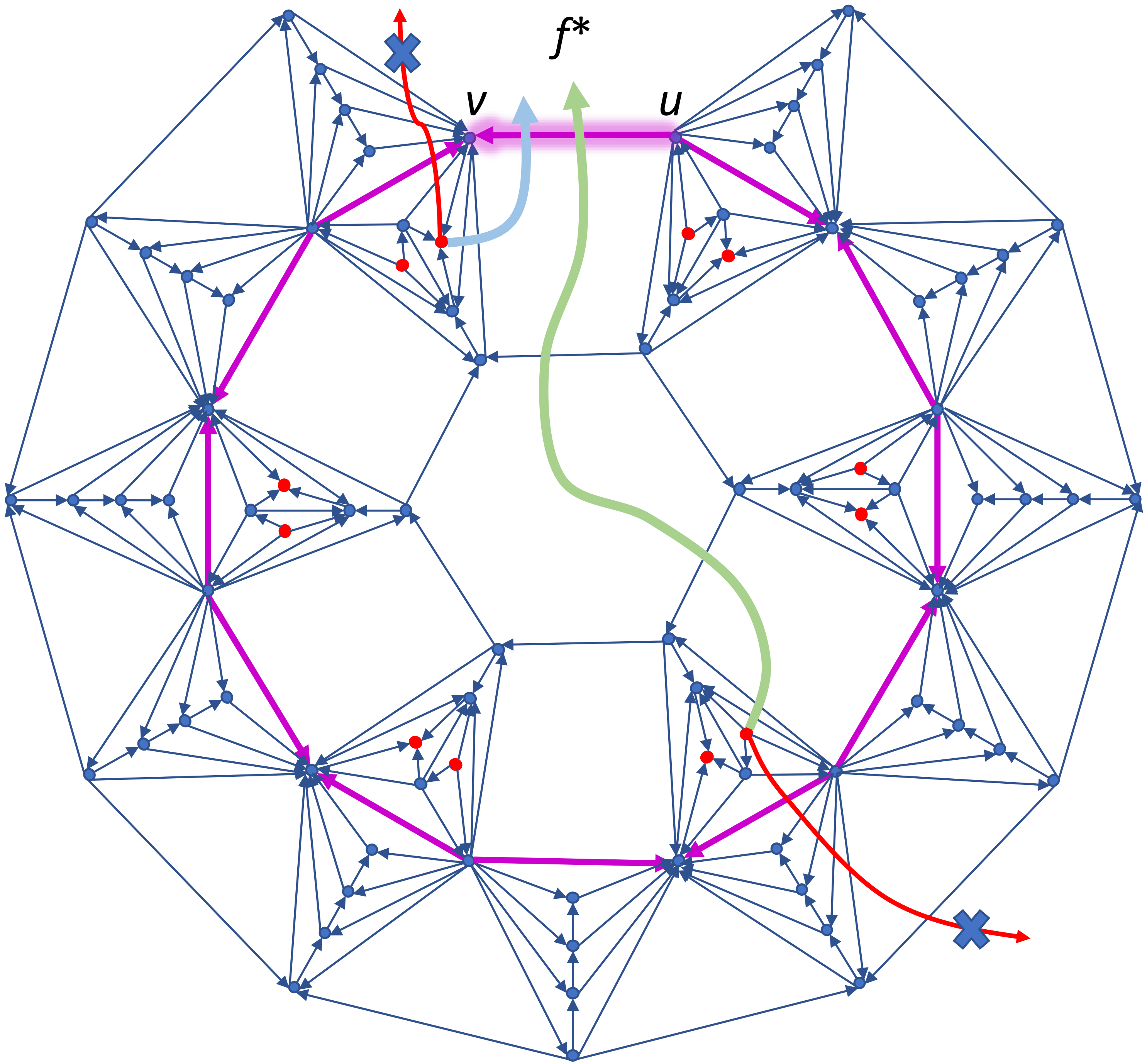


G_6

f^* has capacity at least k-1.

↓

At least k – 1 units of flow must traverse edge (u,v).

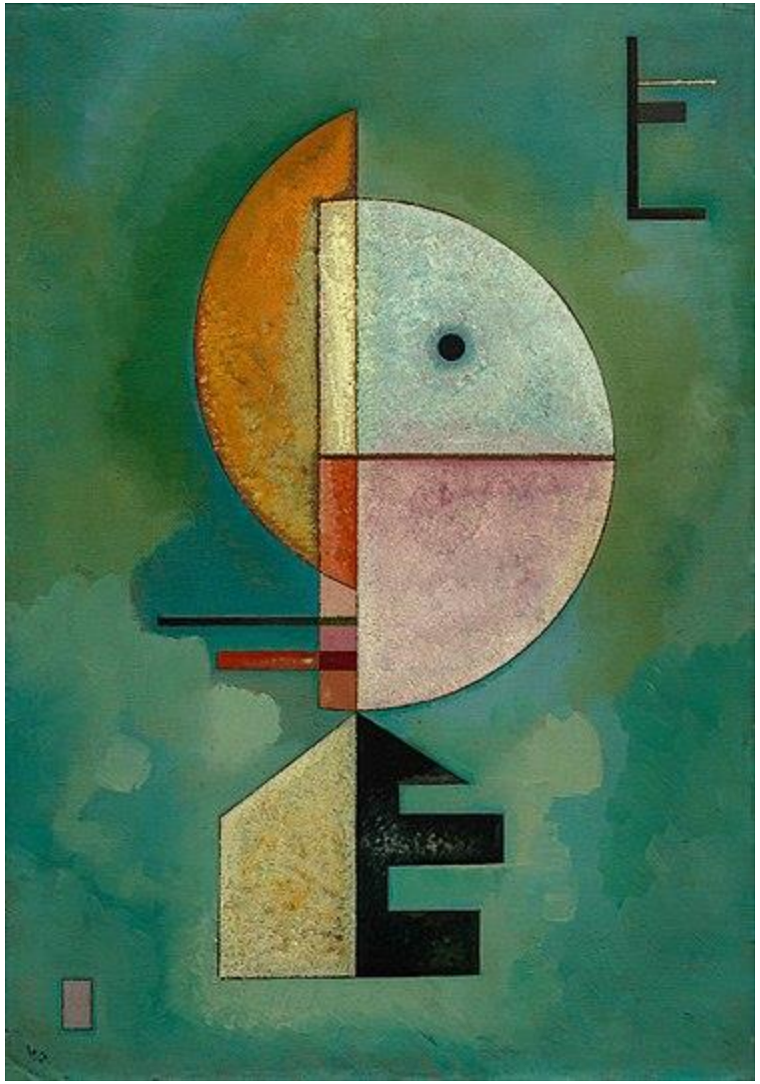


Some open problems

- Is it possible to improve the time complexity needed for computing a quasi-upward planar drawing with curve complexity two?

- It would be interesting to minimize the total number of subdivision vertices (with at most two subdivision vertices per edge) such that the resulting graph admits an upward straight-line drawing of polynomial area.

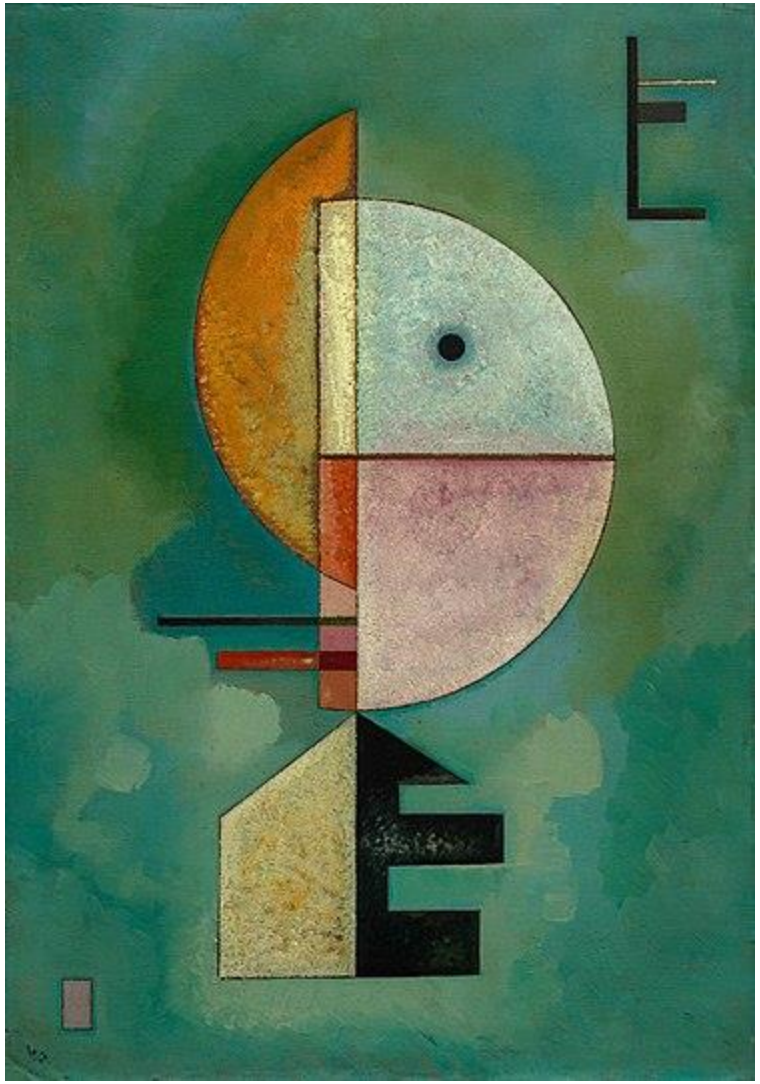
Thank you!



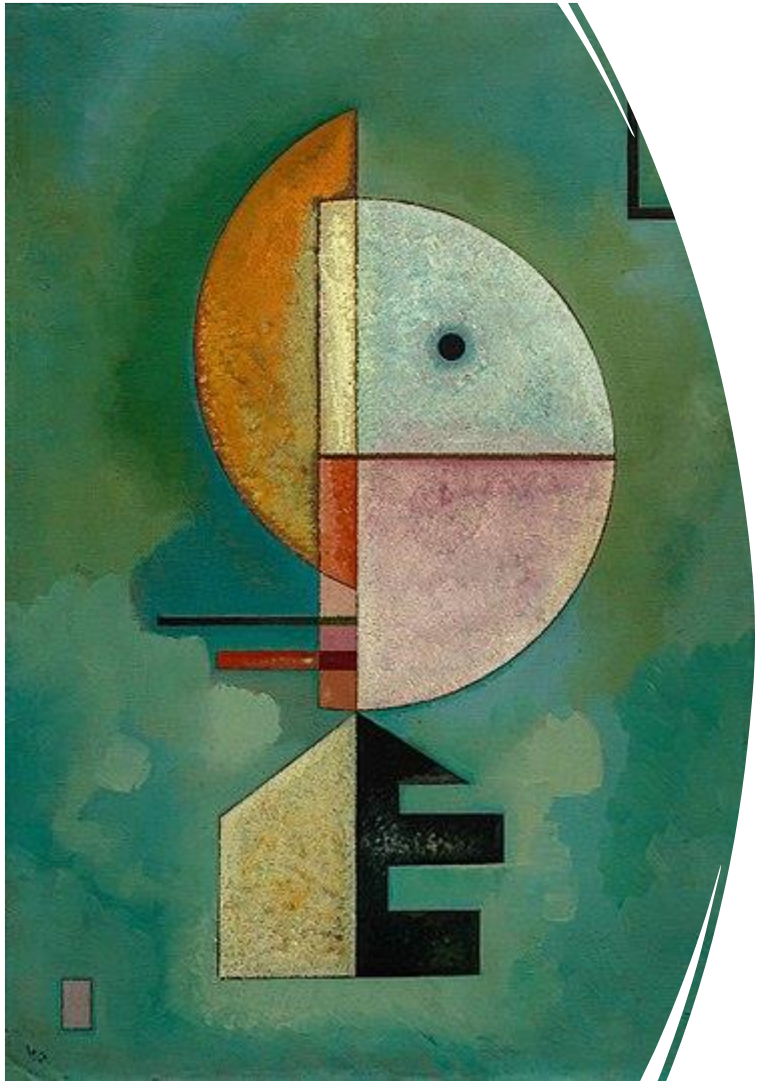
Vasily Kandinsky,"Upward"
Peggy Guggenheim Collection,
Venice

Upward

Thank you!



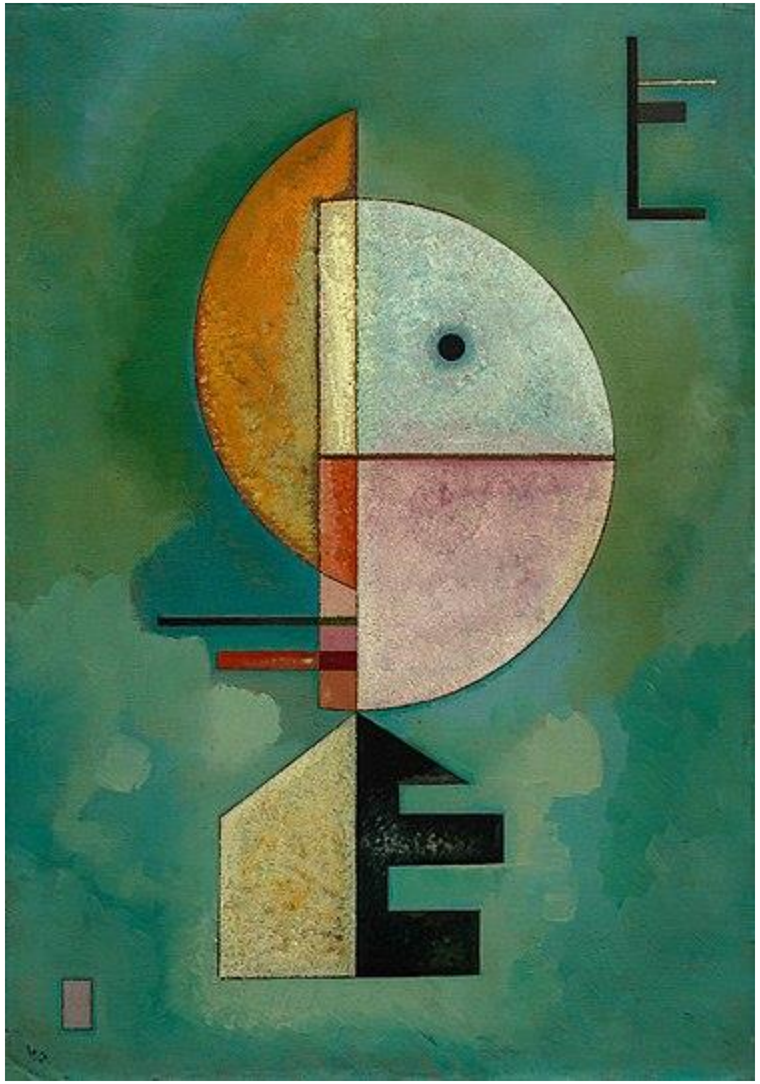
Vasily Kandinsky,"Upward"
Peggy Guggenheim Collection, Venice



Upward

"Quasi "-upward

Thank you!



Vasily Kandinsky,"Upward"
Peggy Guggenheim Collection, Venice

Upward

"Quasi "-upward