Edge-Minimum Saturated k-Planar Drawings

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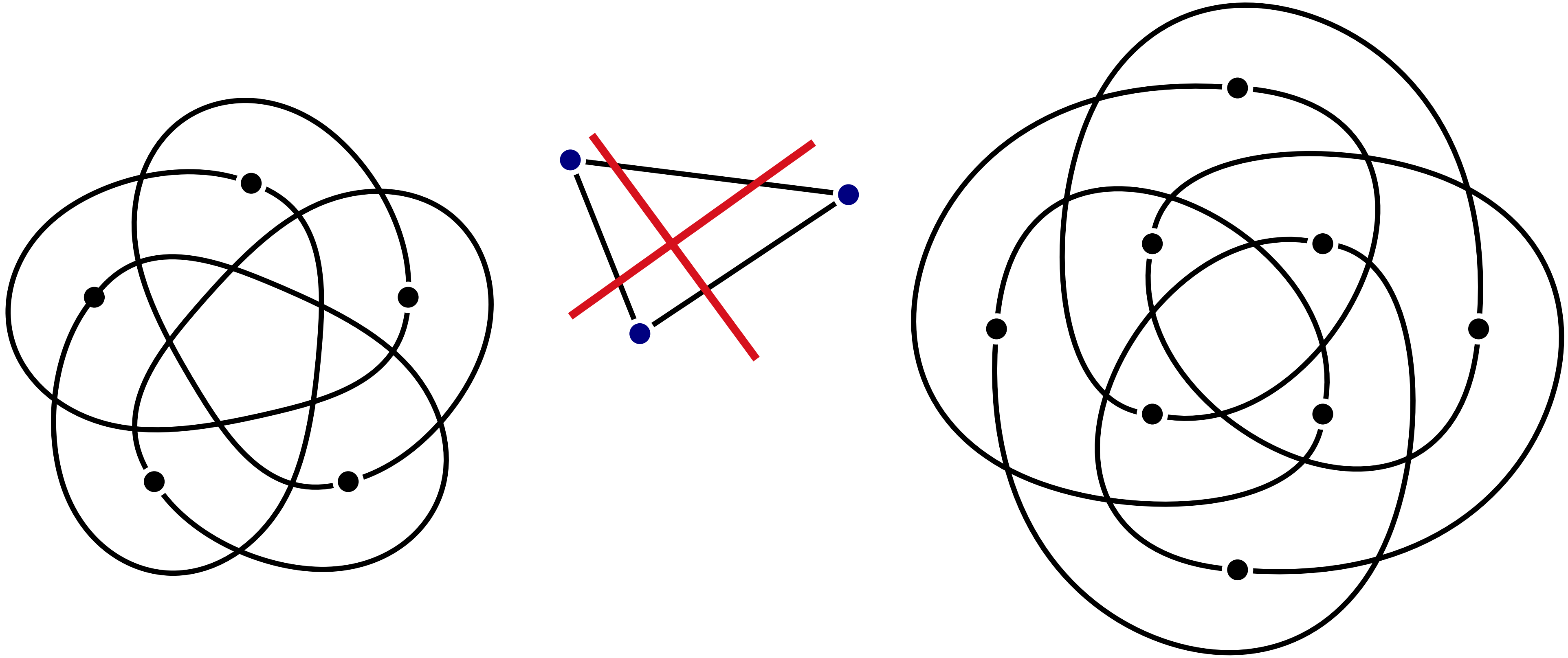
Jonathan Rollin, and Torsten Ueckerdt

Graph Drawing 2021 · September 15, 2021

Karlsruher Institut für Technologie
Maastricht University
DTU
FernUniversität in Hagen
Utrecht University

Saturated drawings

A drawing D is *saturated for a drawing style* Γ if $D \in \Gamma$ and the addition of any edge to D results in a drawing not in Γ.



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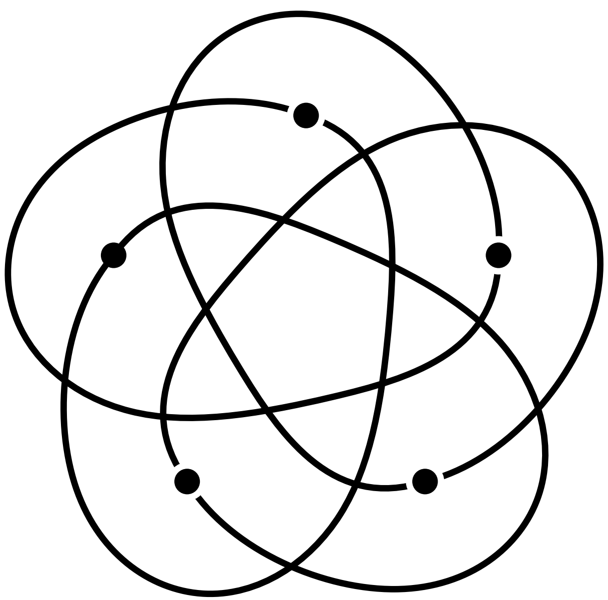
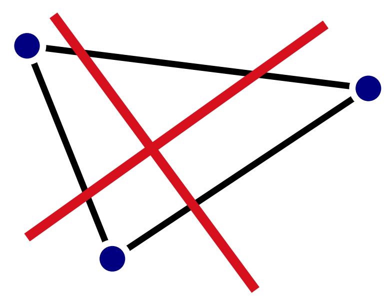
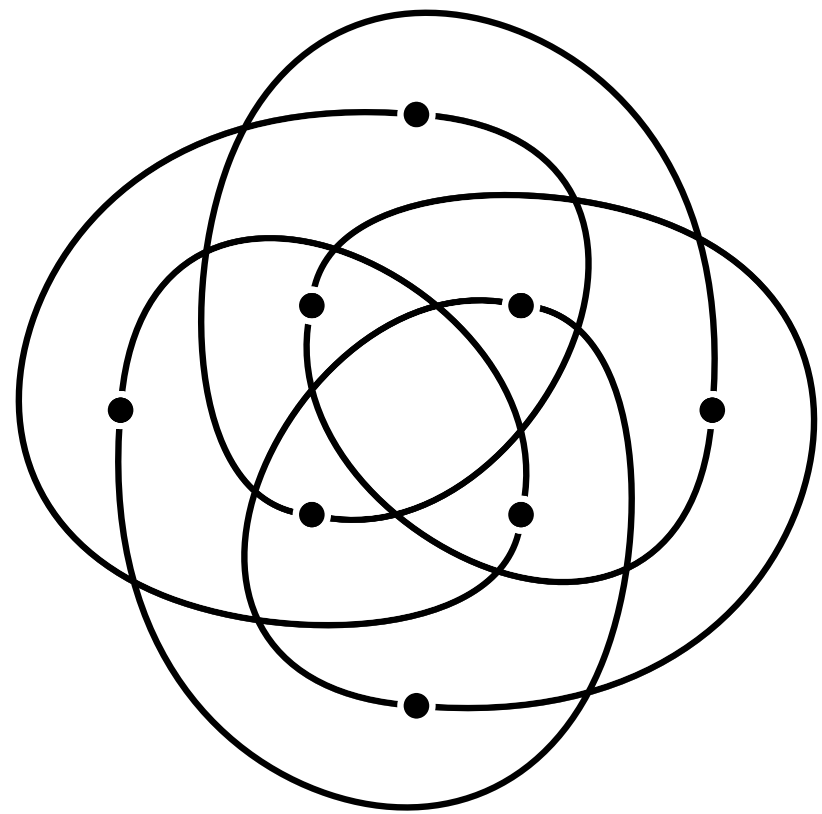




4-planar



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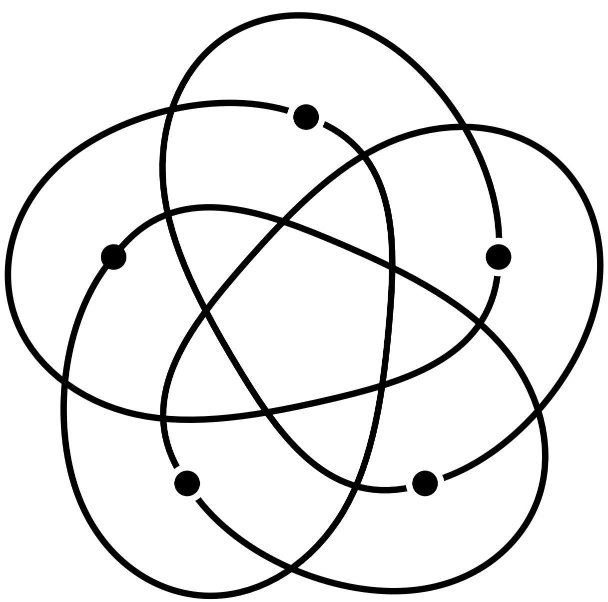

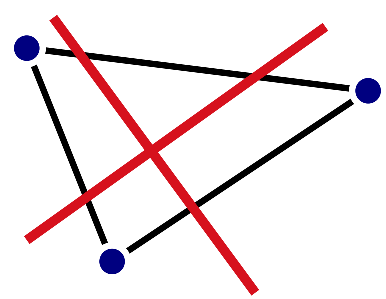

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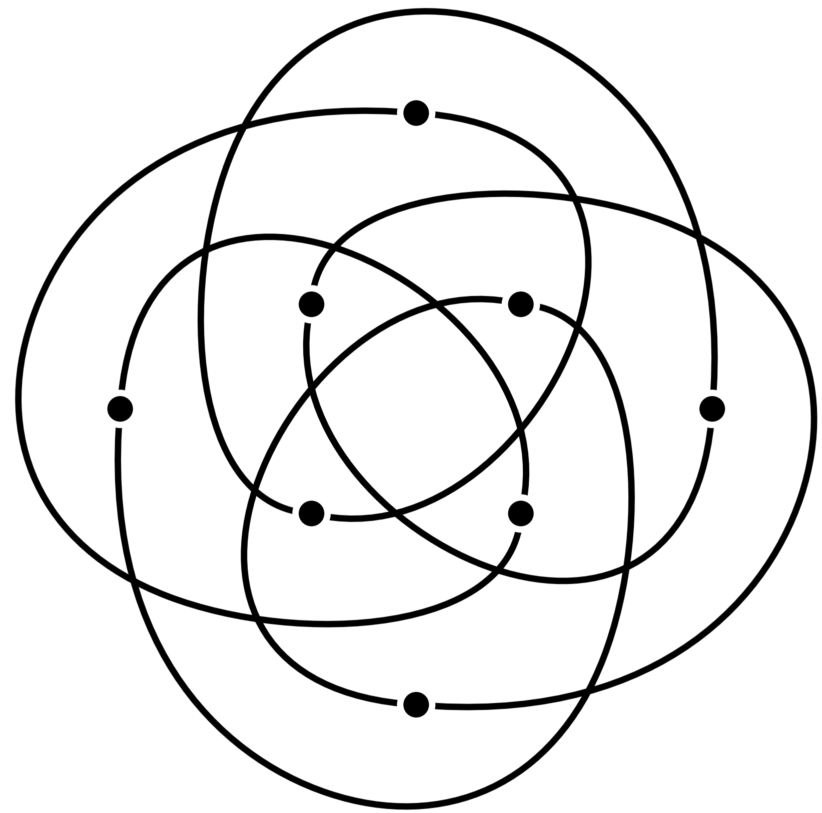

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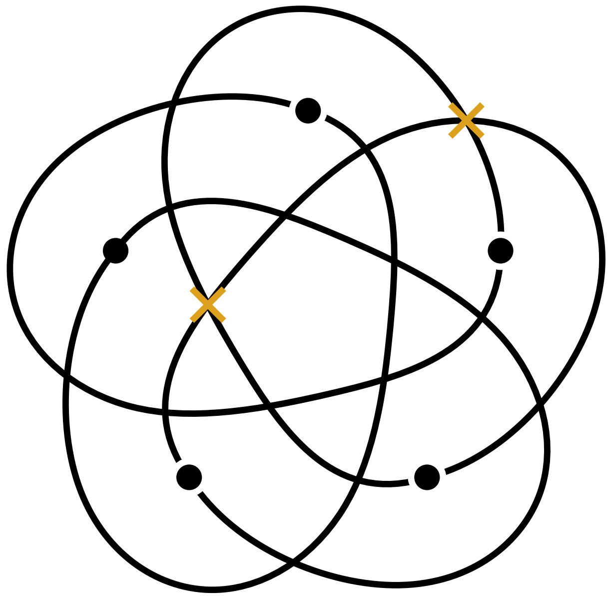

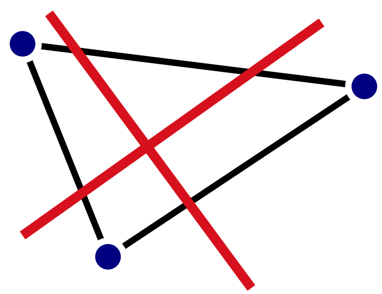

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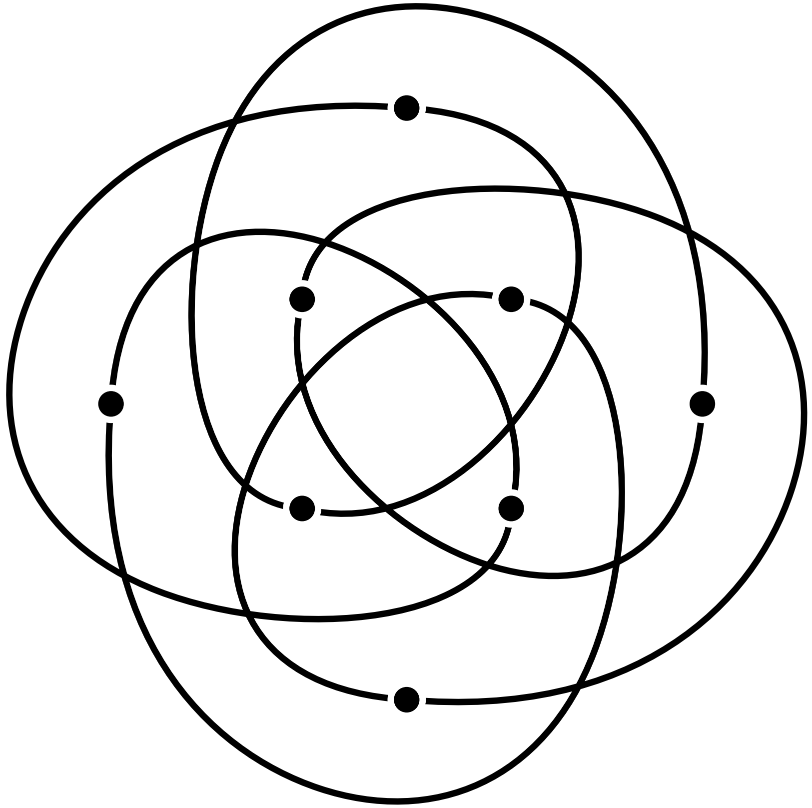

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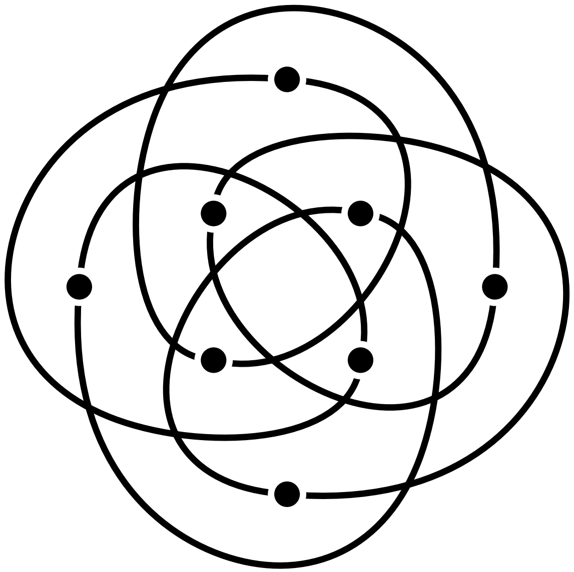

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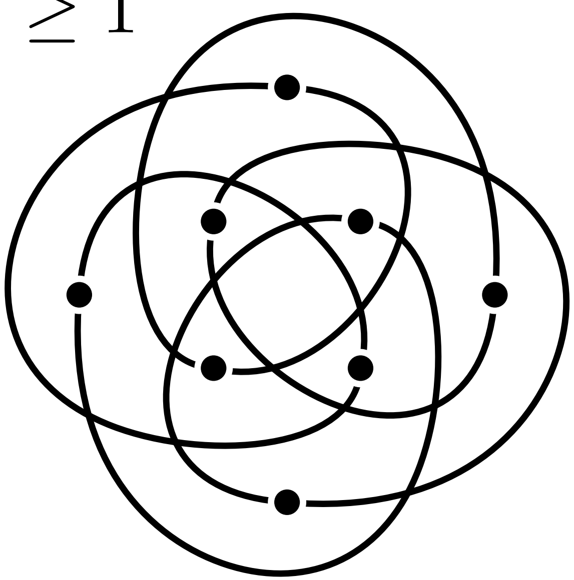
Graph connected? $\Rightarrow$ #edges $\geq n-1 \Rightarrow \alpha \geq 1$

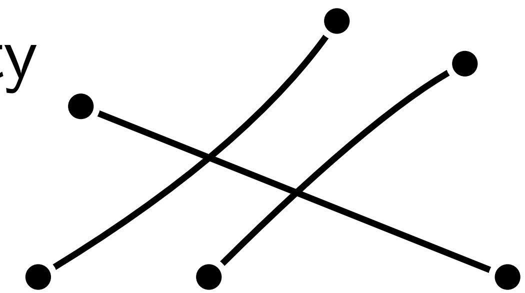
Contiguous drawing? $\Rightarrow \alpha \geq \frac{1}{2}$

For $k \geq 4$? $\Rightarrow \alpha \leq 1$

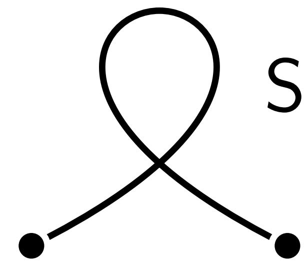


Drawing styles

k-Planarity

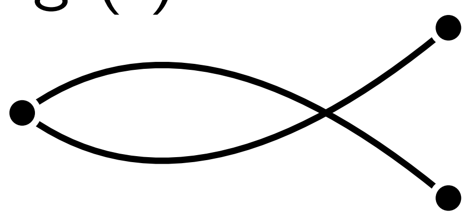


e.g. $k = 2$



Self-crossing (**S**)

Incident-crossing (**I**)



Multicrossings (**M**)

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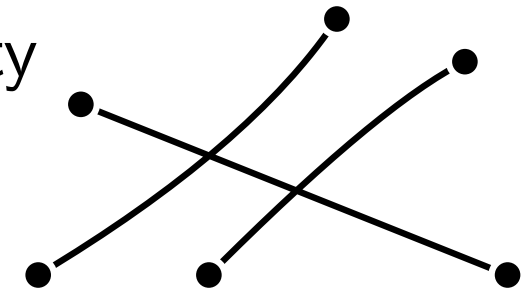

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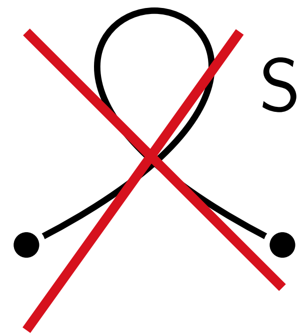

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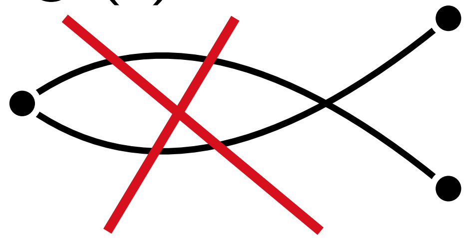
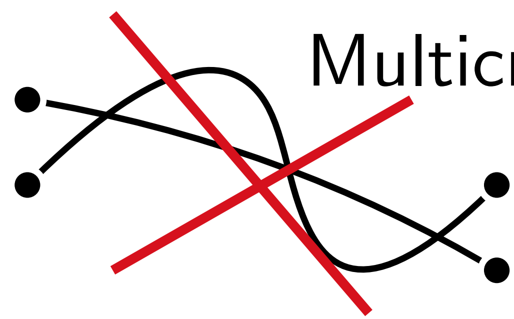
Incident-crossing (**I**) -free





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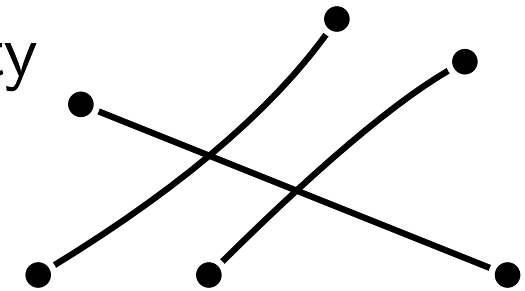

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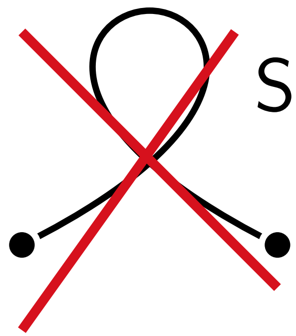

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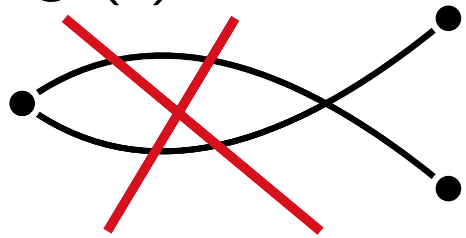

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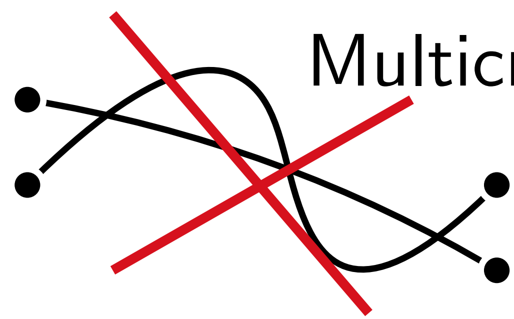

Note: drawing style $\{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ = simple/good drawings

Saturation problems

Max-saturated (aka optimal) and min-saturated

Both directions studied for abstract graphs without K_t

Max [Turán'41]; Min [Erdős, Hajnal, and Moon'64]

For planar graphs they coincide: $3n-6$ edges

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Max-saturated k-planar

k-planar + simple drawing $\Rightarrow \leq 3.81\sqrt{k}n$ edges

[Ackermann'19]

1-planar + simple drawing $\Rightarrow 4n - 8$ edges

[Pach and Tóth'97]

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Min-saturated outisde k-planarity

Simple drawings [Hajnal et al.'18] & [Kynčl et al.'15]

. . .

Saturated drawings $\times$ styles

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For example: no selfcrossings $\{\textbf{S}\}$ and 4-planar

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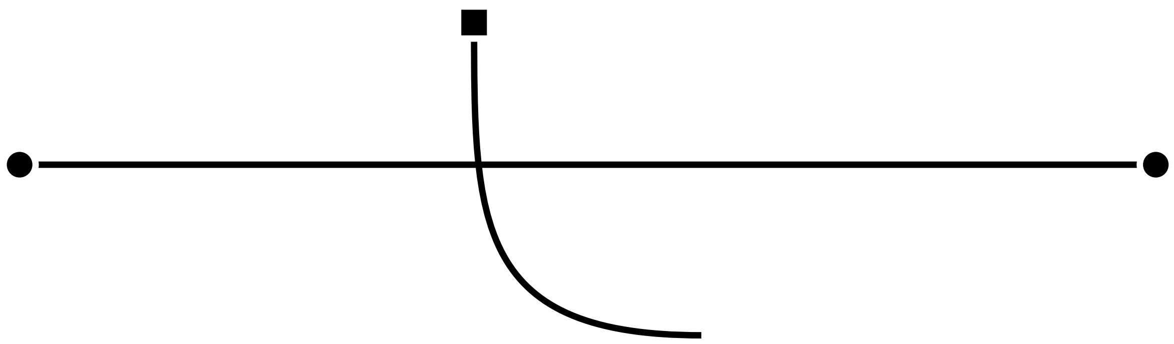

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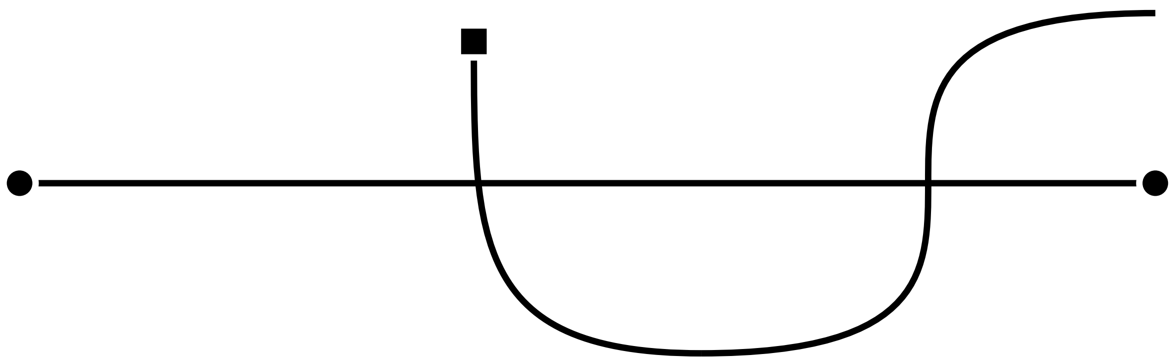

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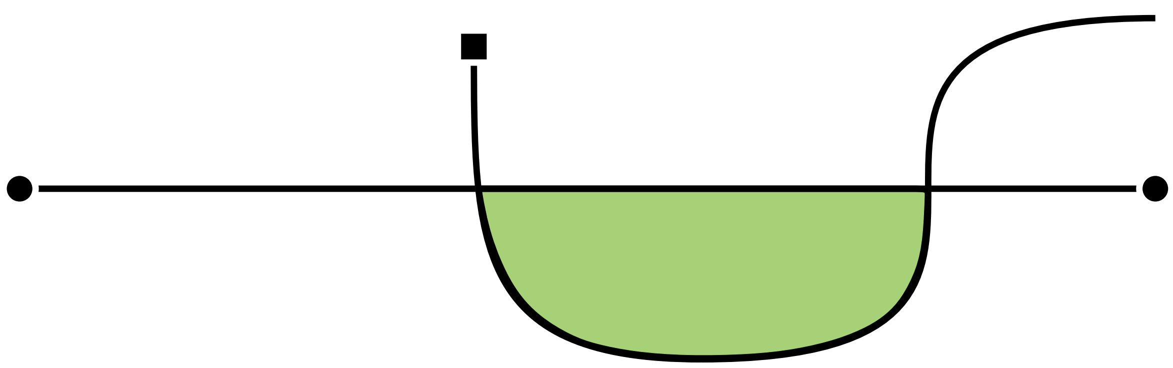

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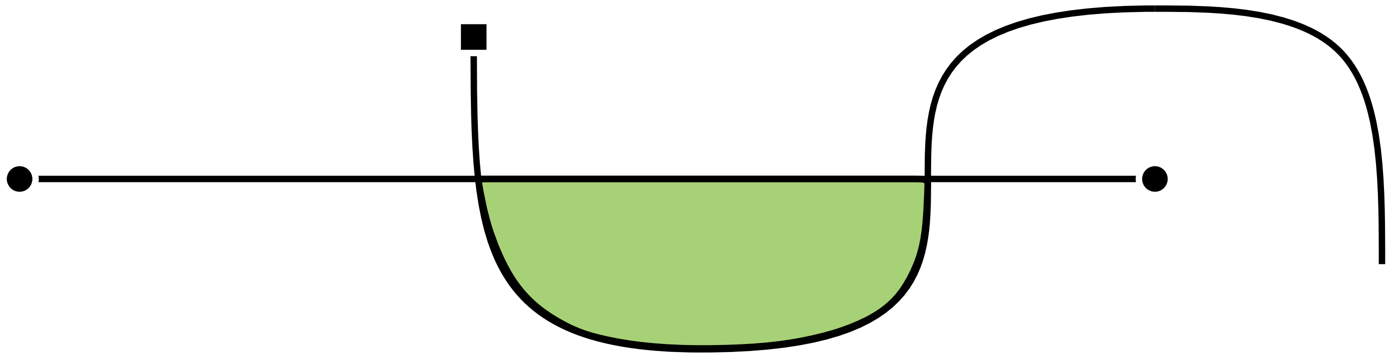

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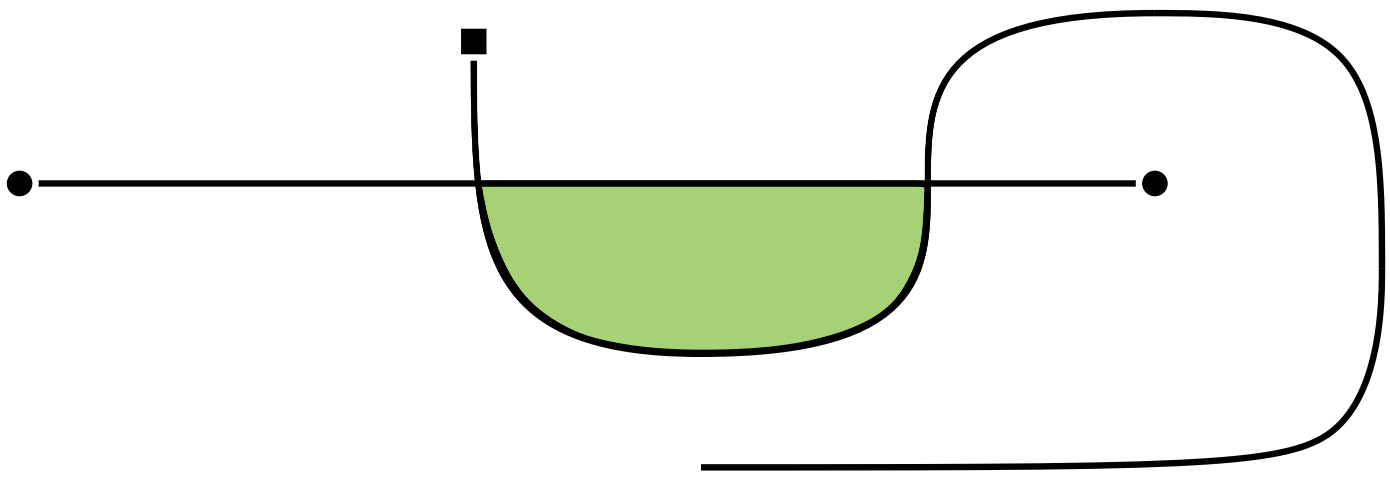

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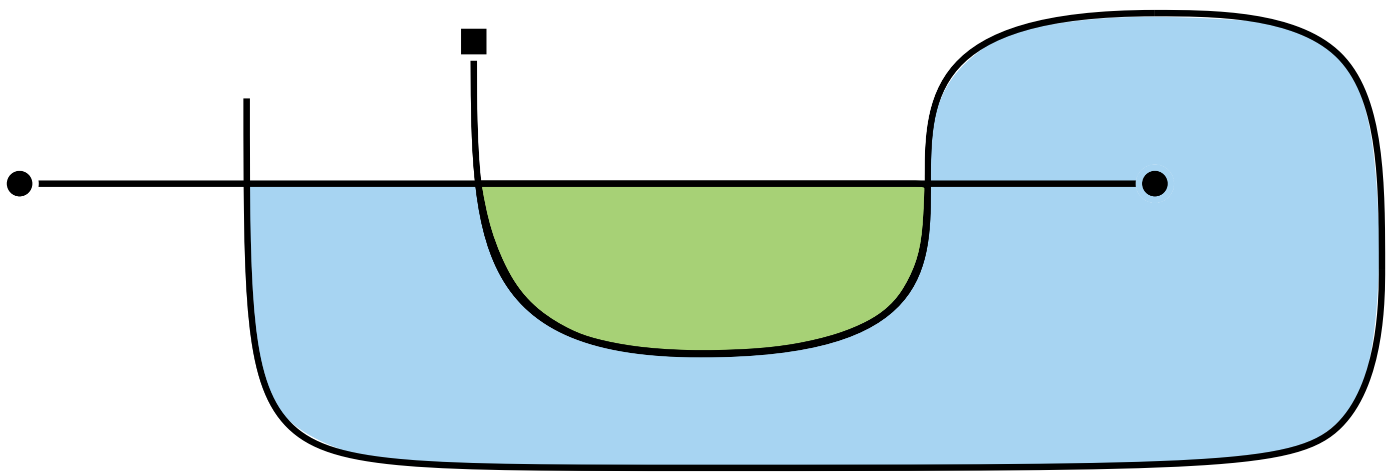

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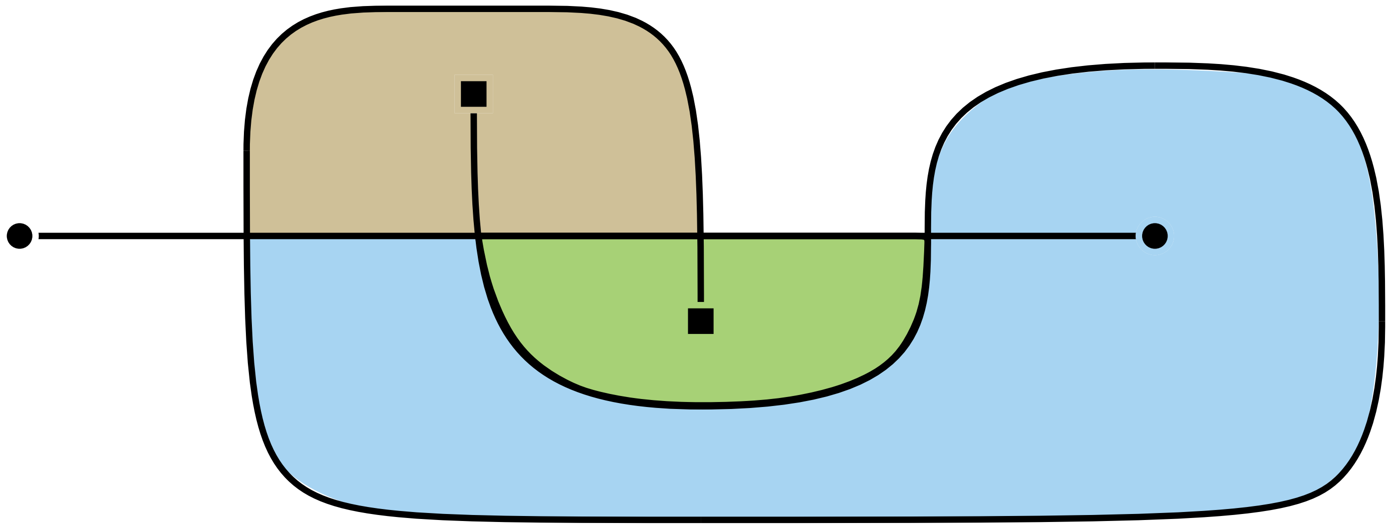

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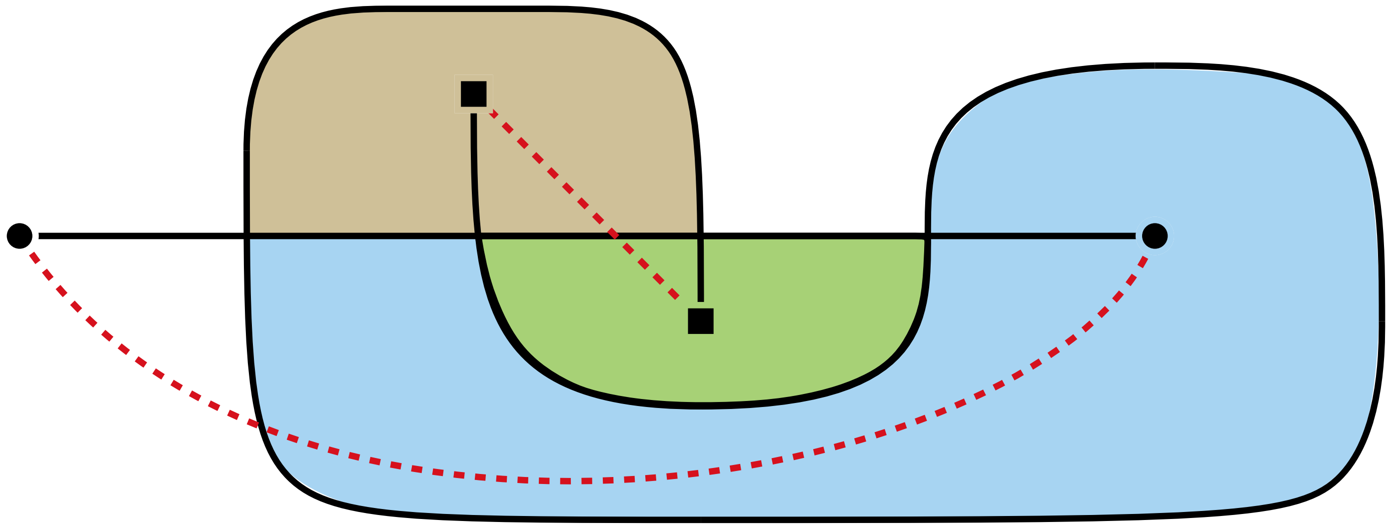

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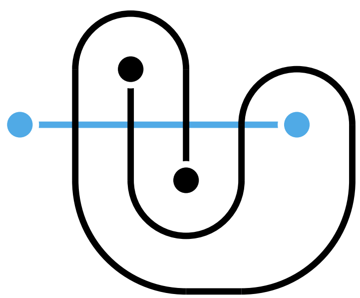

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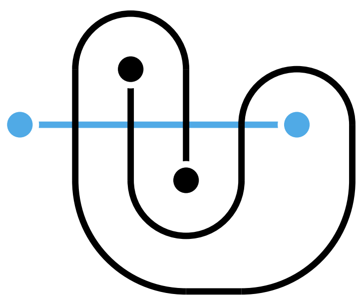
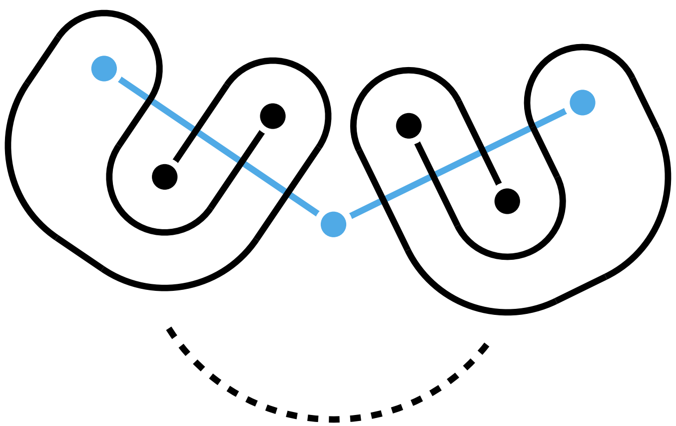


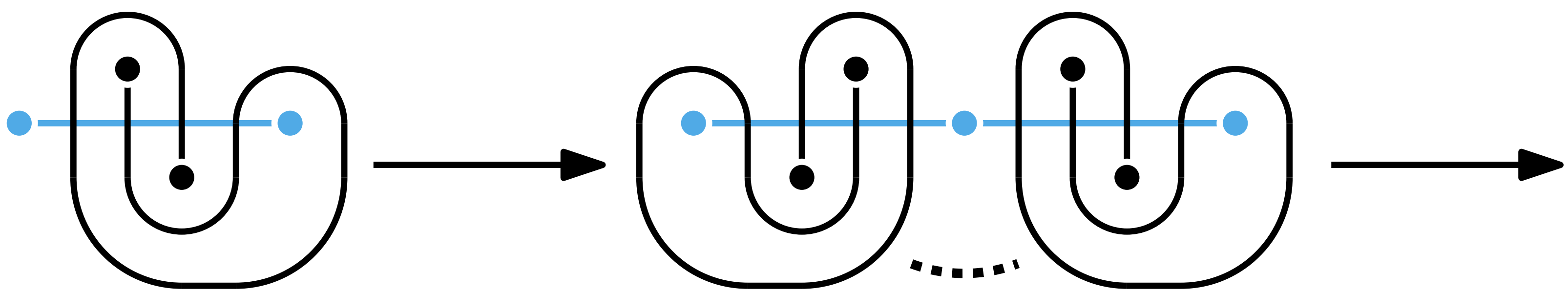


$\Rightarrow \alpha \leq \frac{2}{3}$ for $k = 4$ and $\{\mathbf{S}\}$

General strategy

For n vertices the number of edges will be $\alpha \cdot (n-1) \rightarrow$ find α

Small saturated drawing T in style Γ



Upper bound

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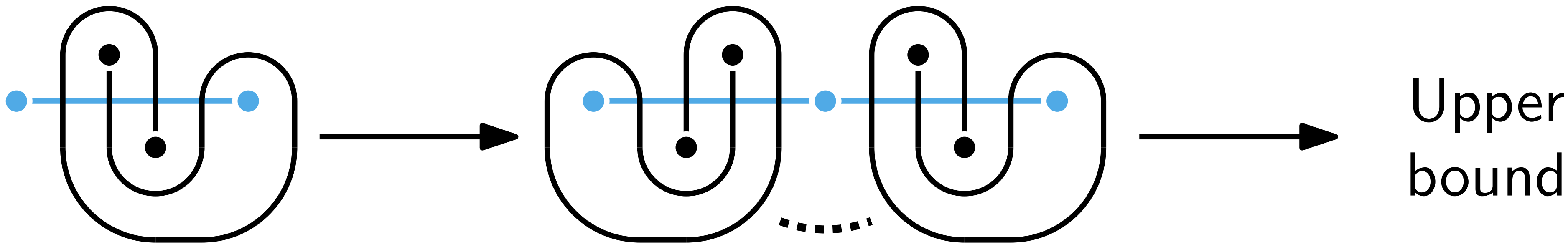

For the lower bound:

Lower bound for **monotone** and **filled** drawings
$\hookrightarrow$ Relies on filled and **essentially-2-connected** drawings

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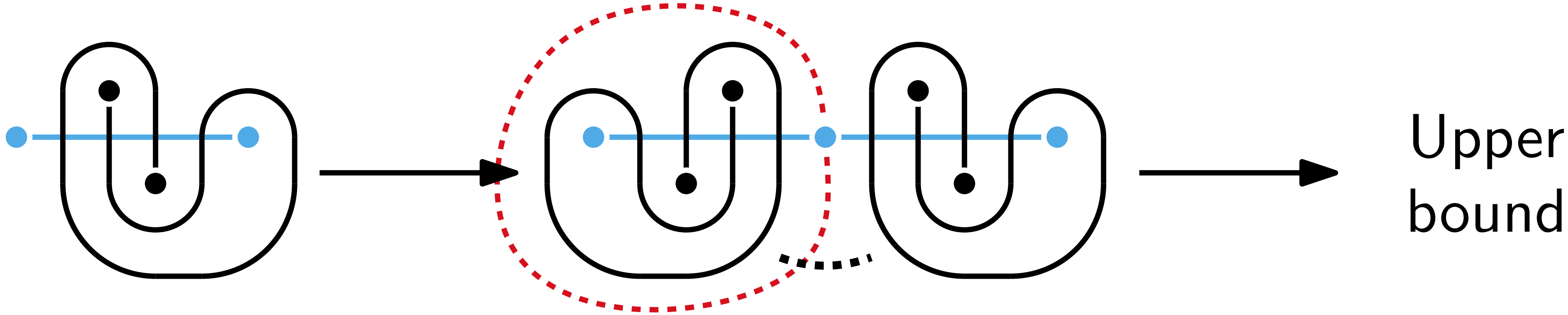

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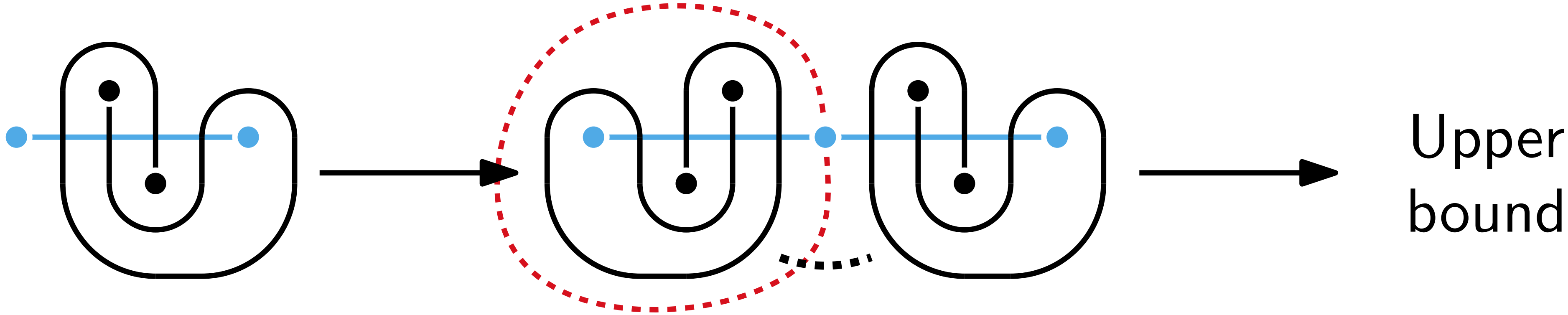

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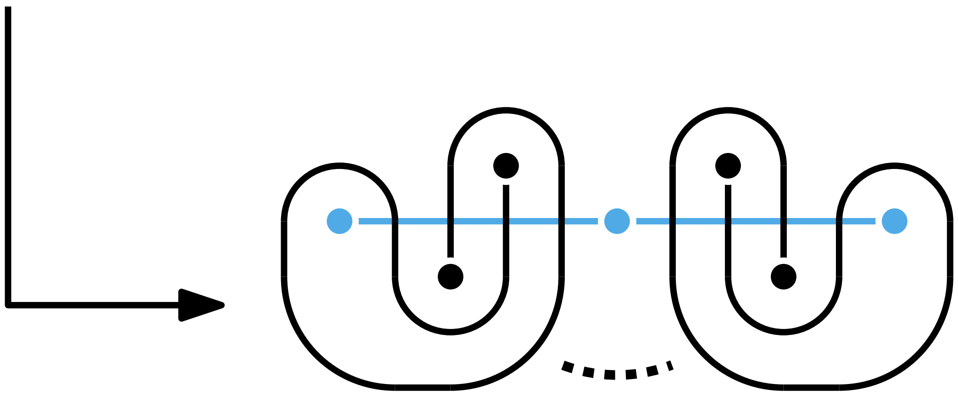
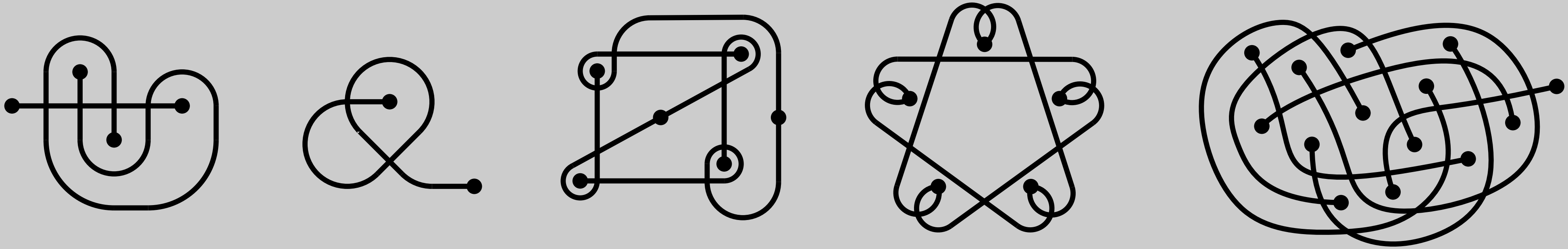
Compute #edges for filled and essentially-2-connected drawings

Tight bound

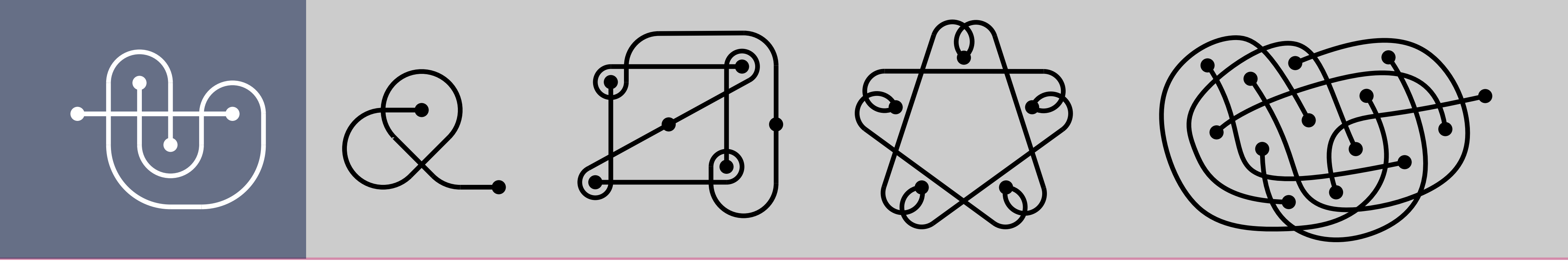
$\alpha = \frac{2}{3}$ for saturated 4-planar drawings and style $\{\mathbf{S}\}$.



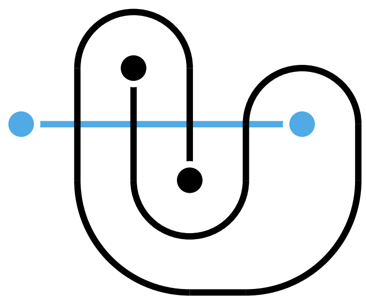
This is the sparsest drawing in this style



Meet the drawings



No selfcrossings (and no incident crossings) $\alpha = \frac{2}{k-1}$



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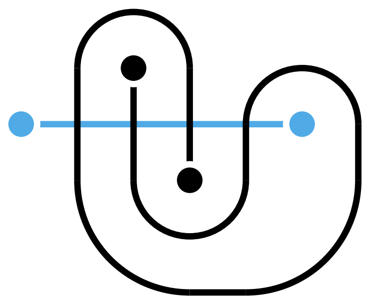

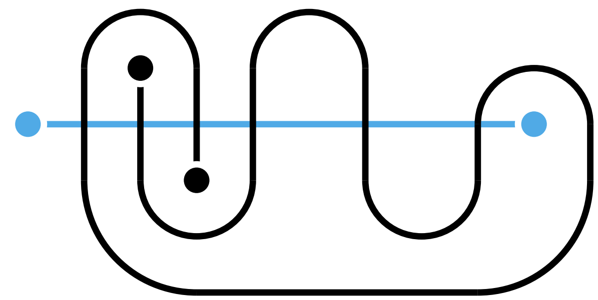

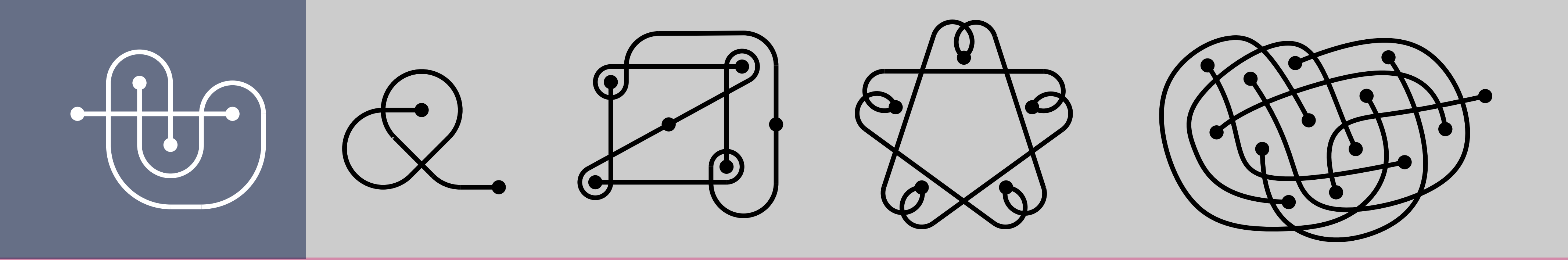

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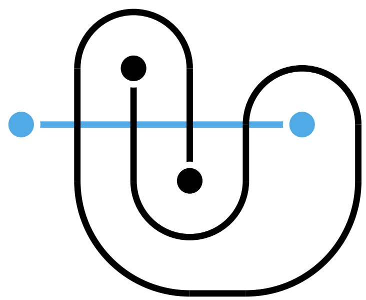

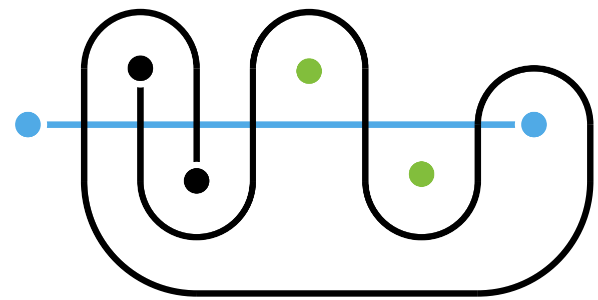

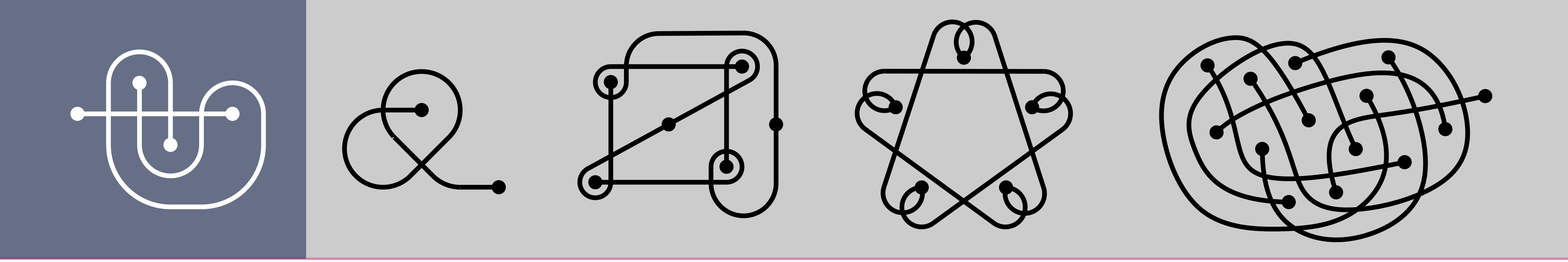

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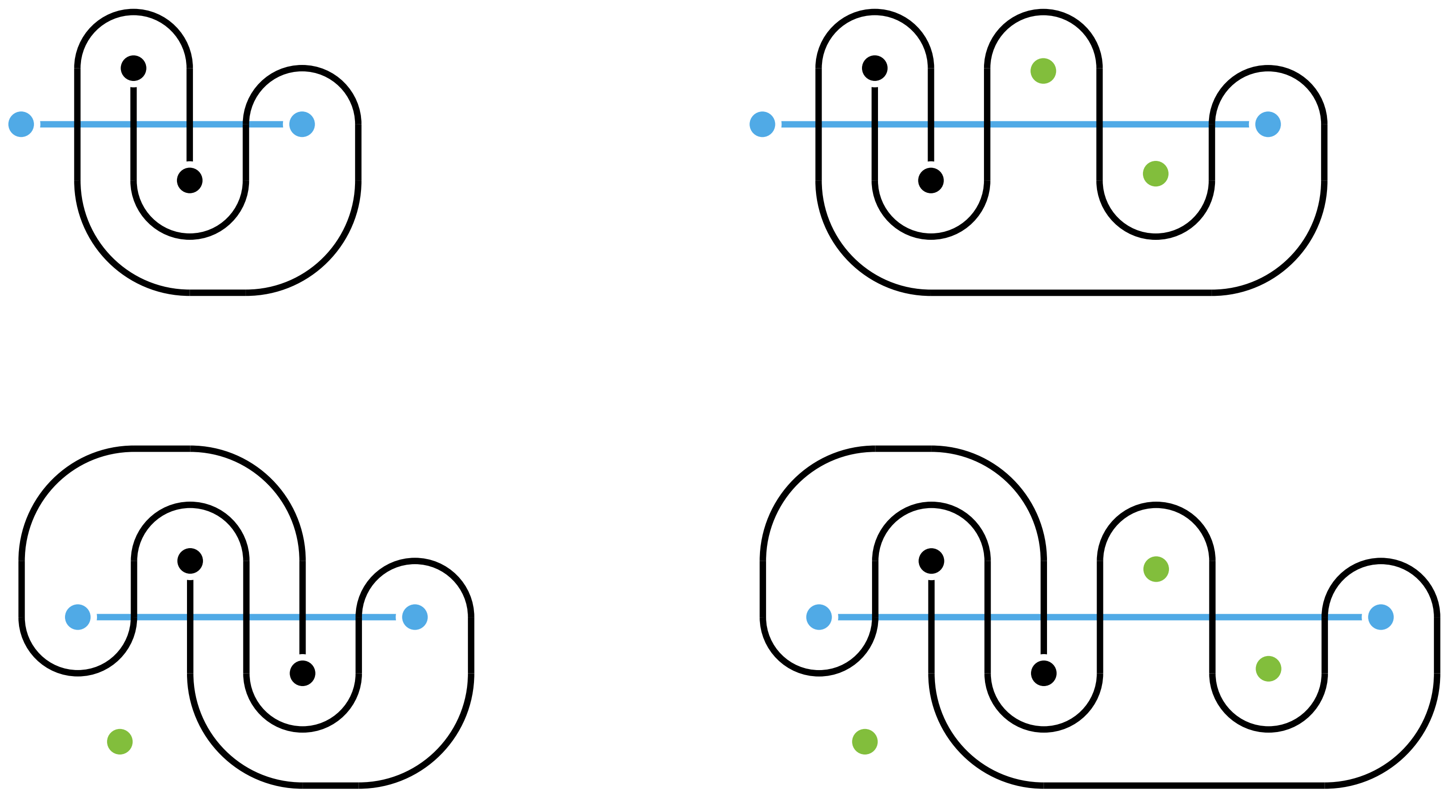
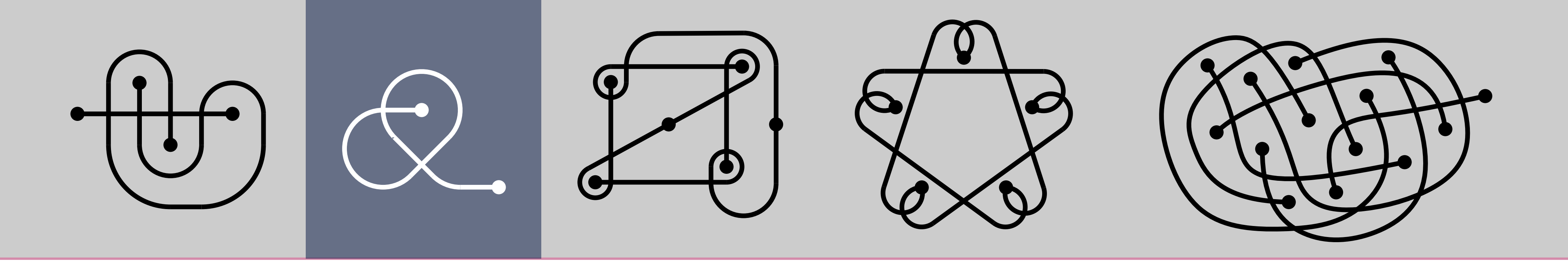
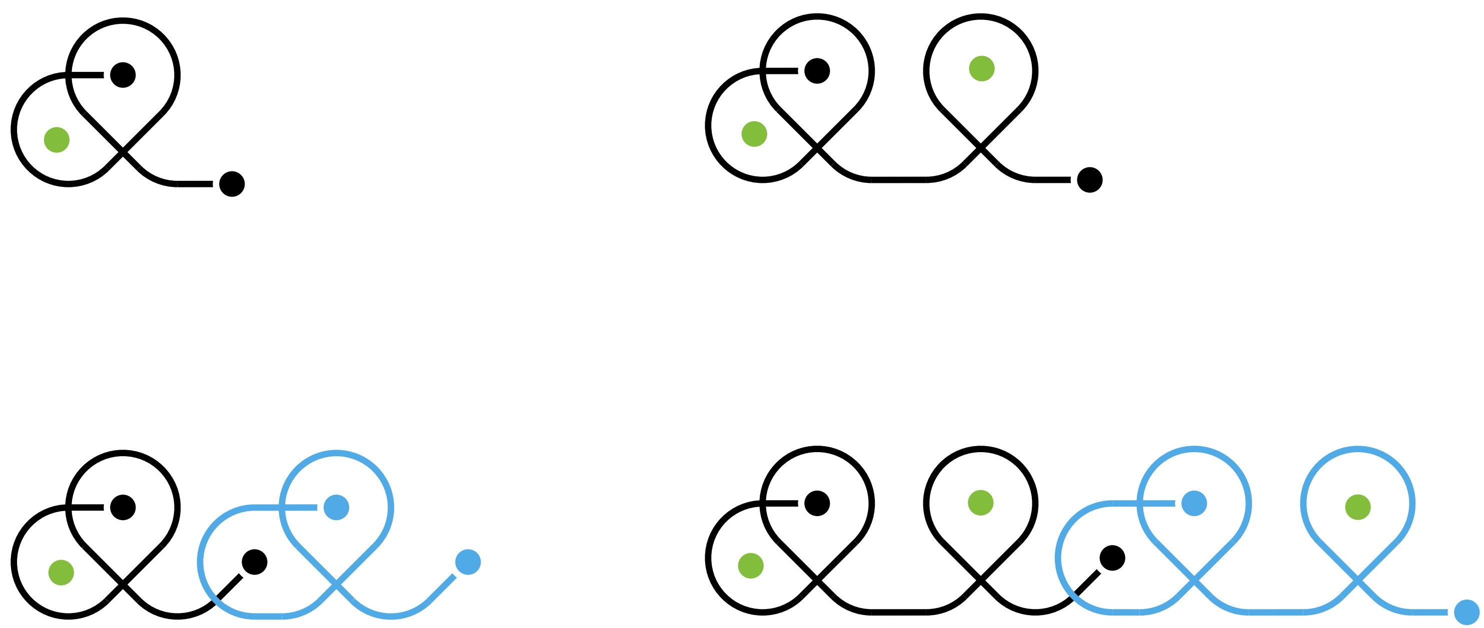
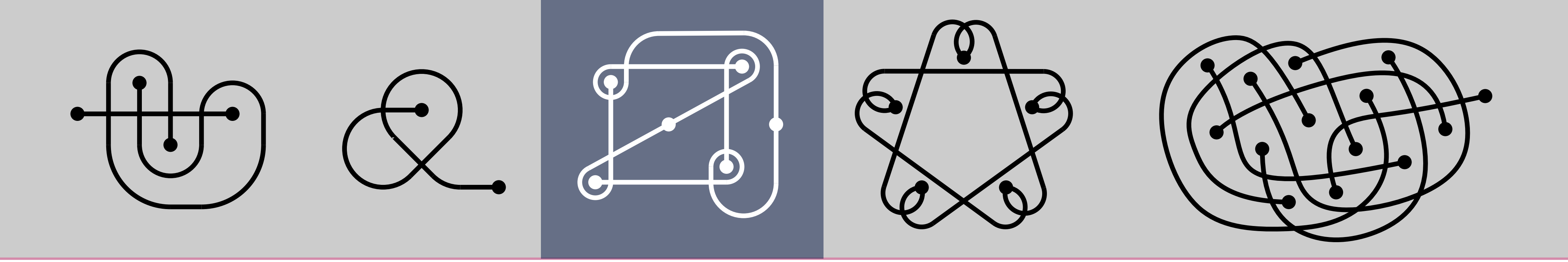




No restrictions or no incident crossings

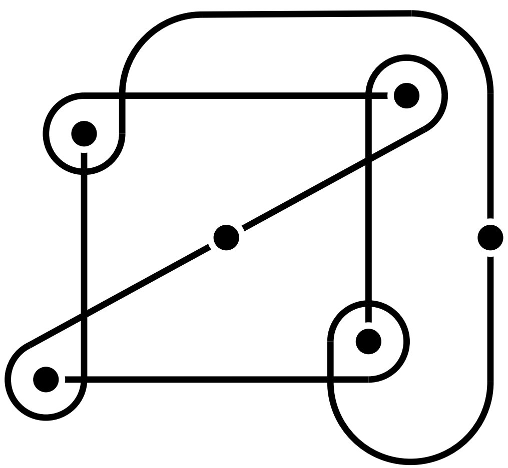
$$\alpha = \frac{2}{k-(k \mod 2)}$$



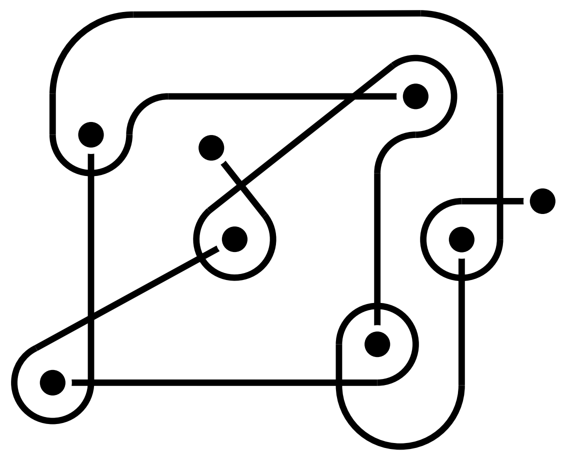


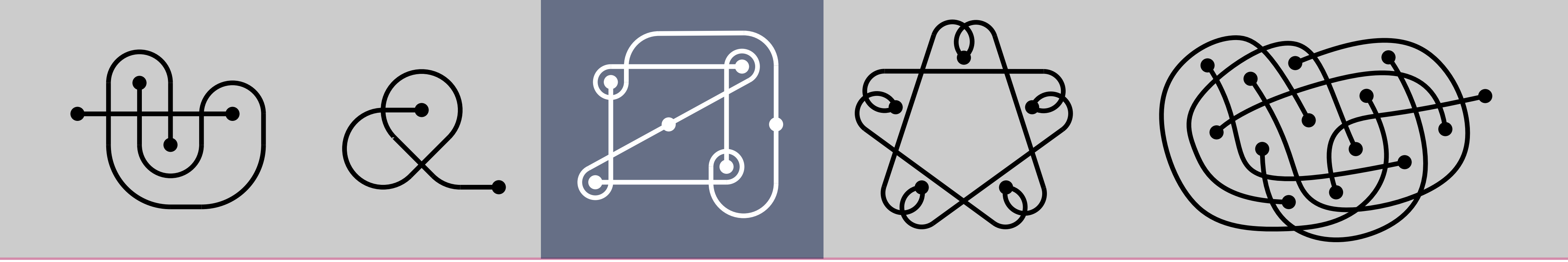
No incident- and no multicrossings

$$\alpha = \frac{2(k-1)}{(k-1)(k-2)+2}$$



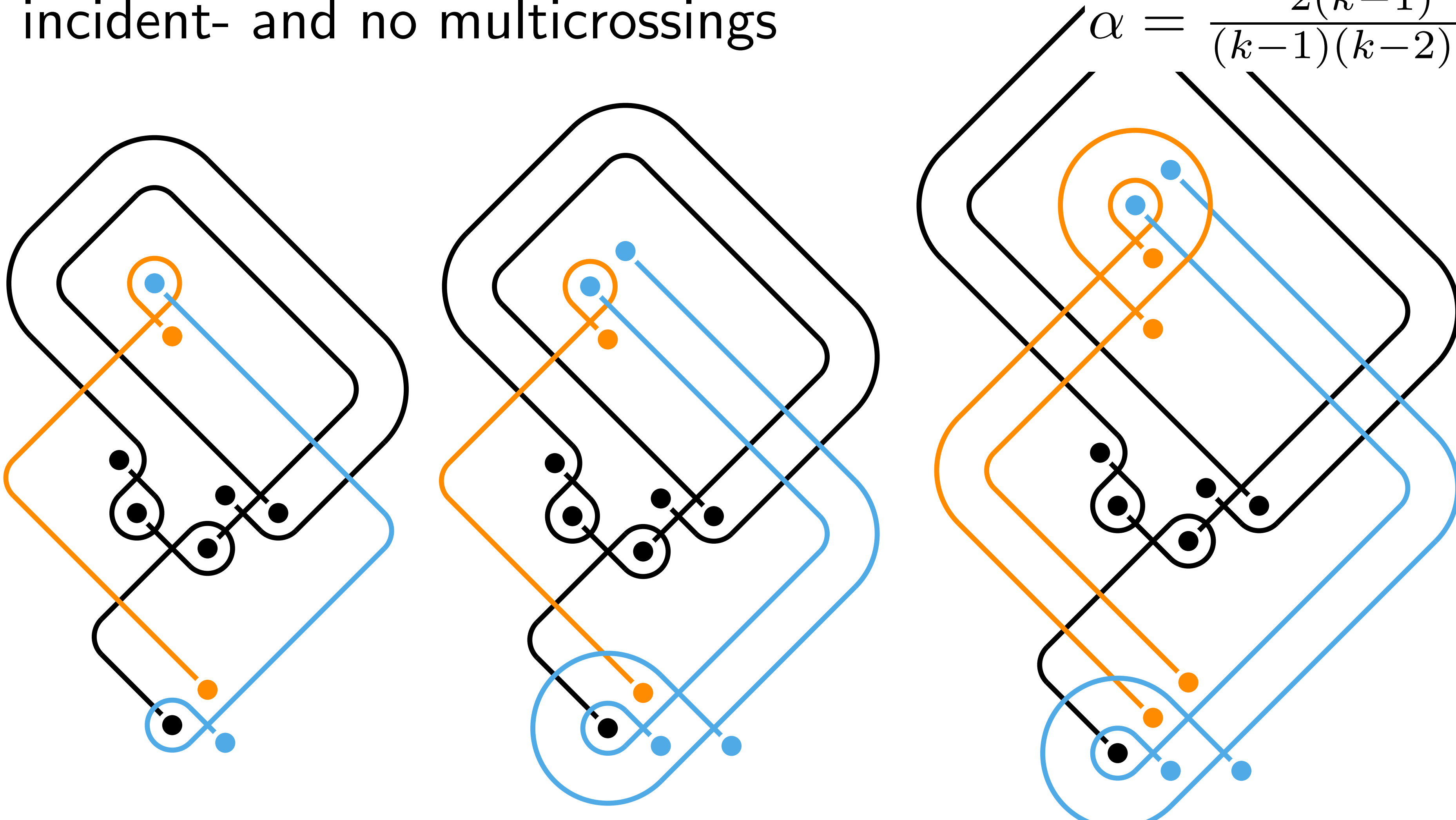
$$\alpha = \frac{4}{5}$$





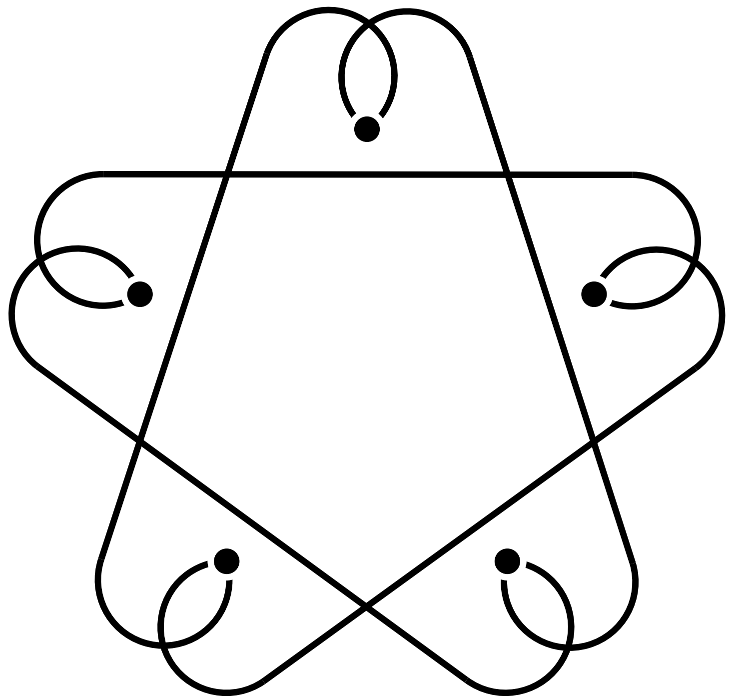
No incident- and no multicrossings

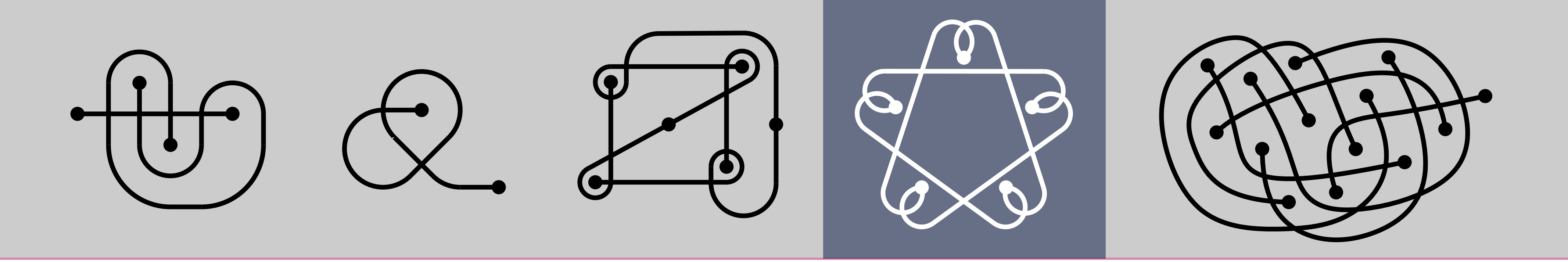
$$\alpha = \frac{2(k-1)}{(k-1)(k-2)+2}$$



No self- and no multicrossings

$$\alpha = \frac{2(k+1)}{k(k-1)}$$





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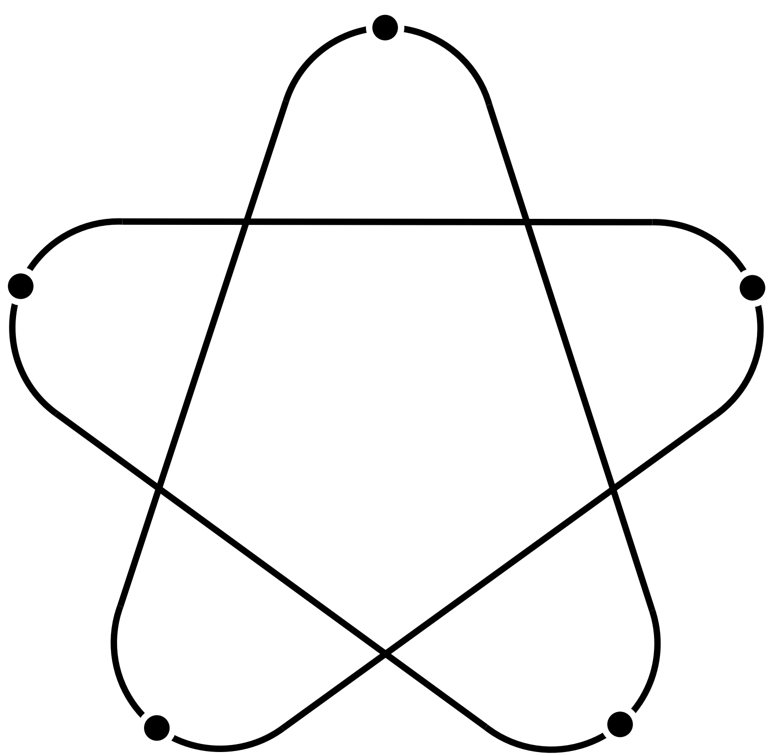

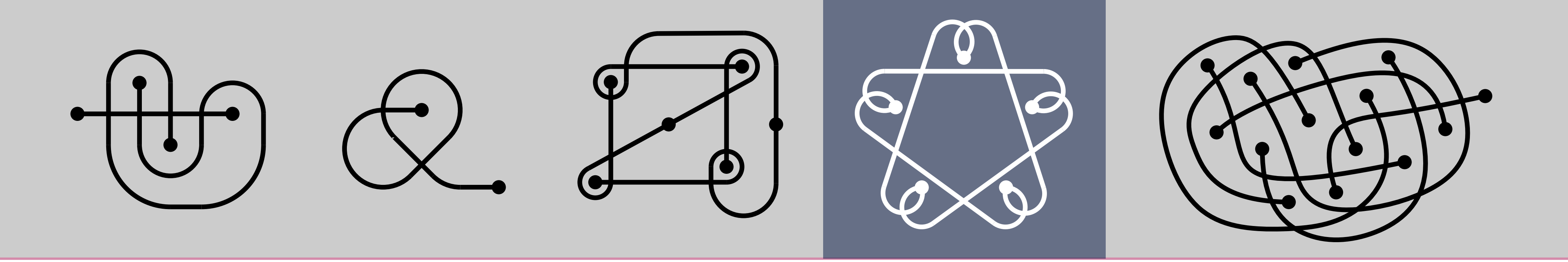

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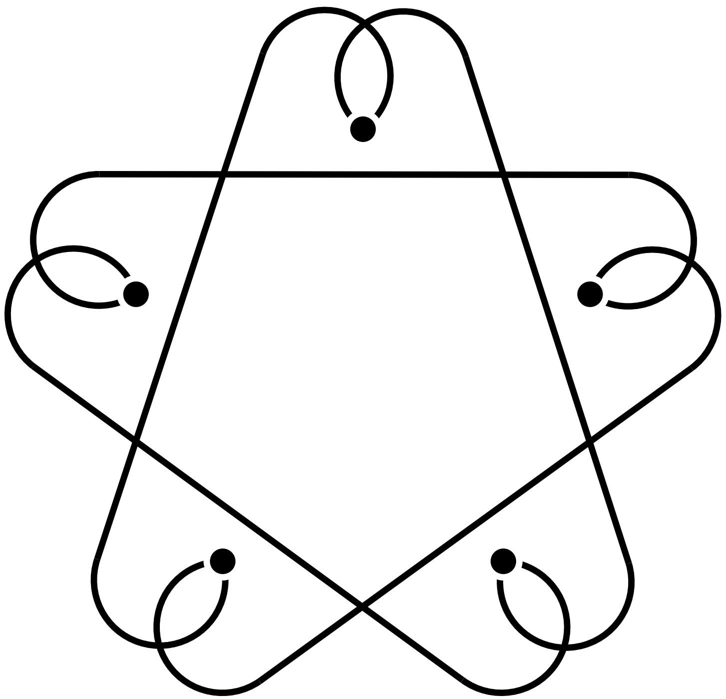

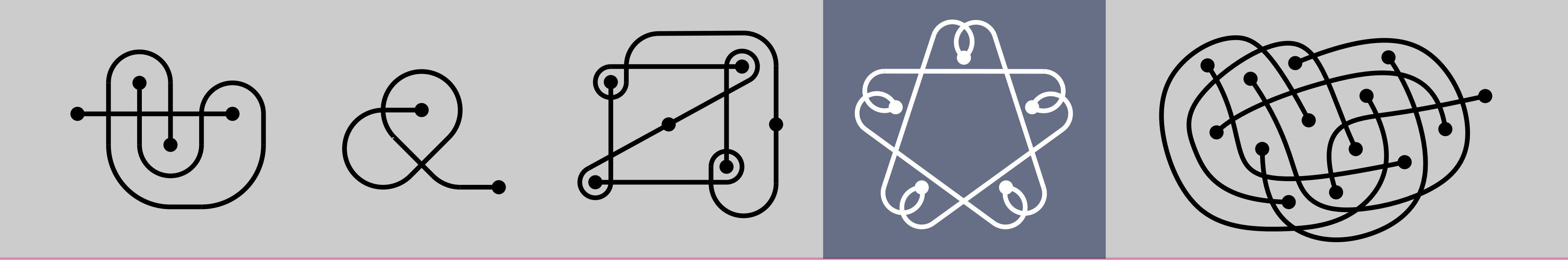

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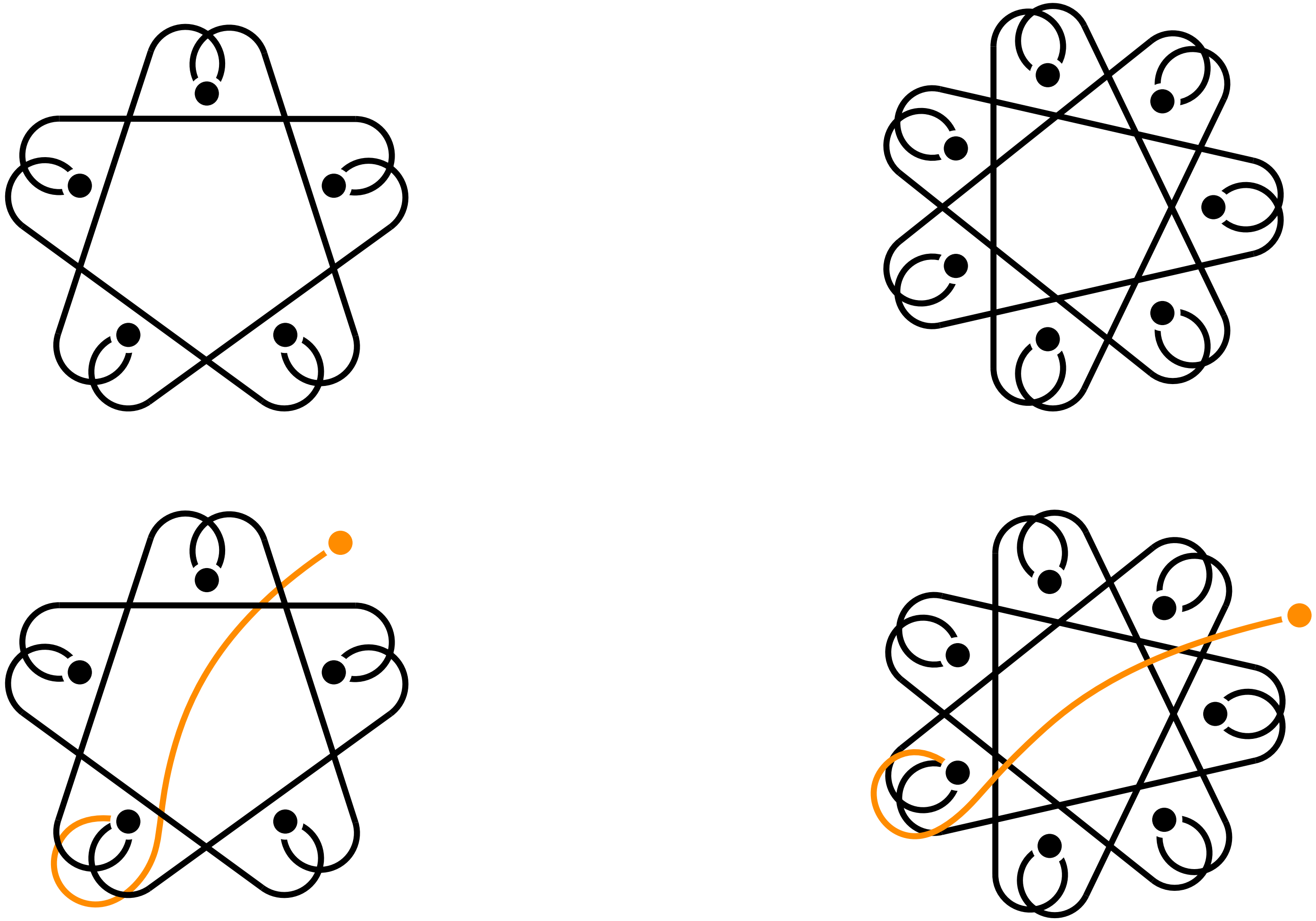
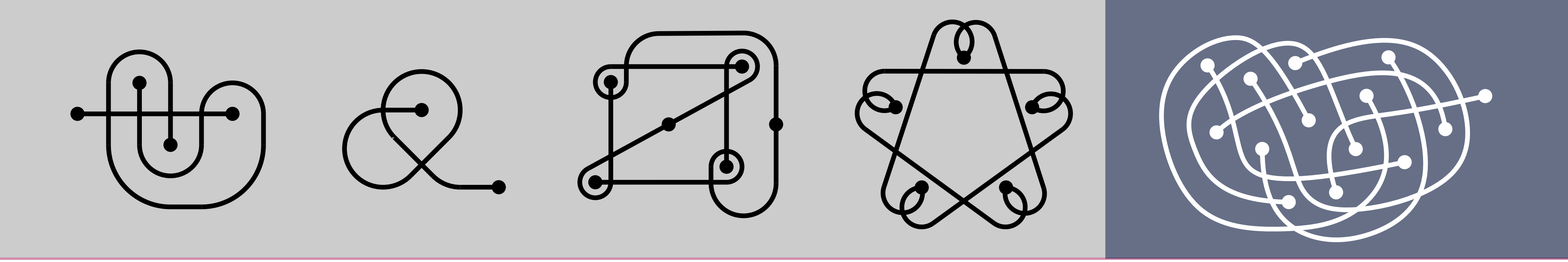
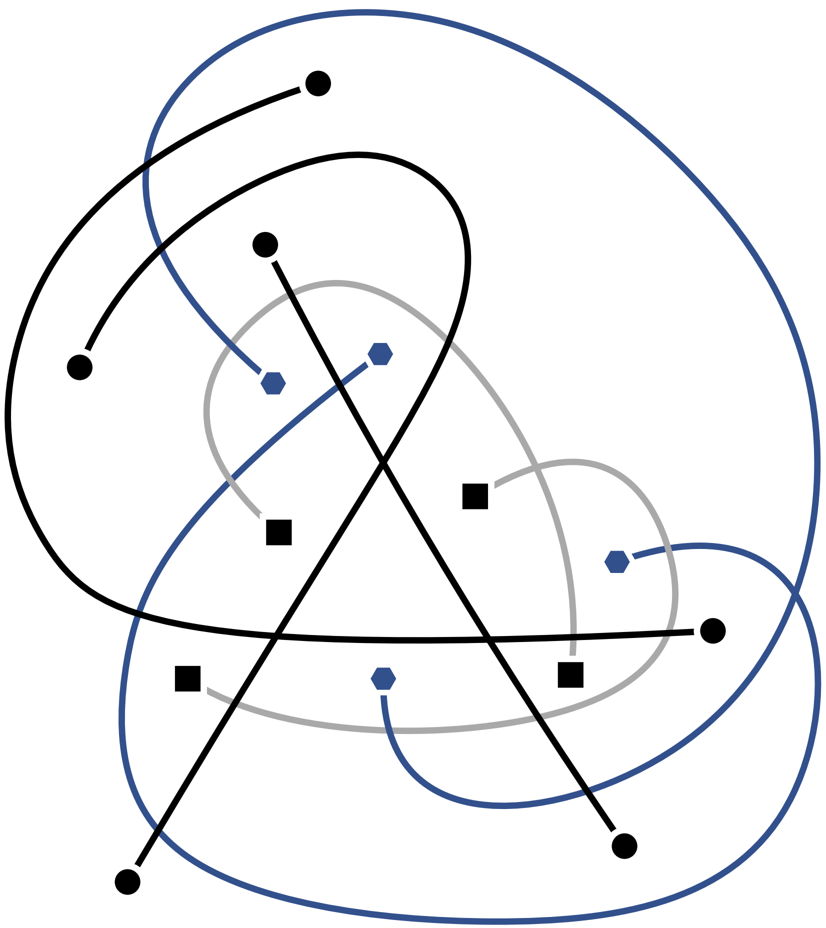




No self-, no incident-, and no multicrossings

$$\alpha = \frac{2(k+1)}{k(k-1)}$$



Only for $k \geq 6$

No self-, no incident-, and no multicrossings

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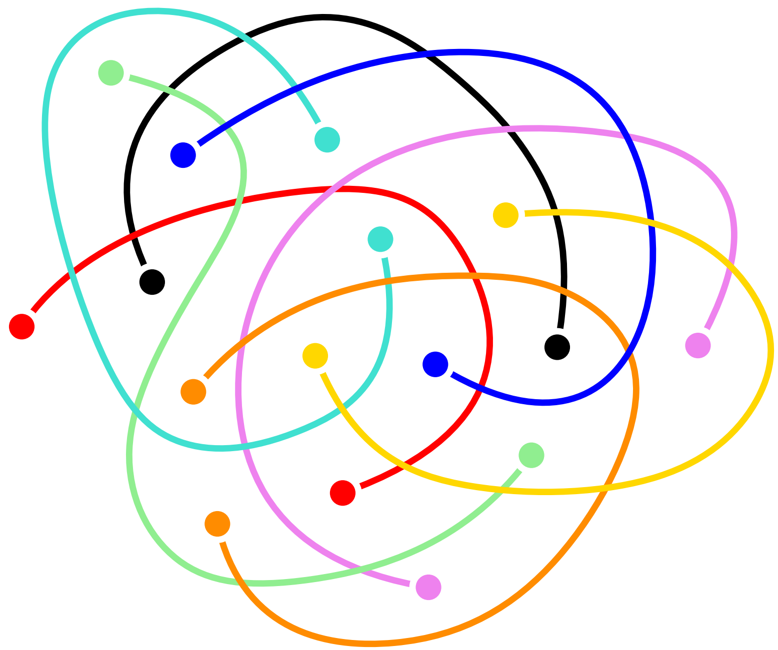

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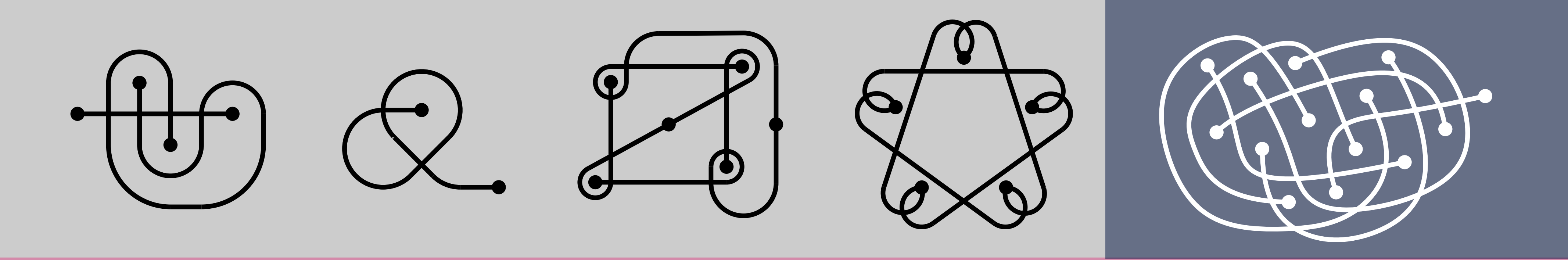

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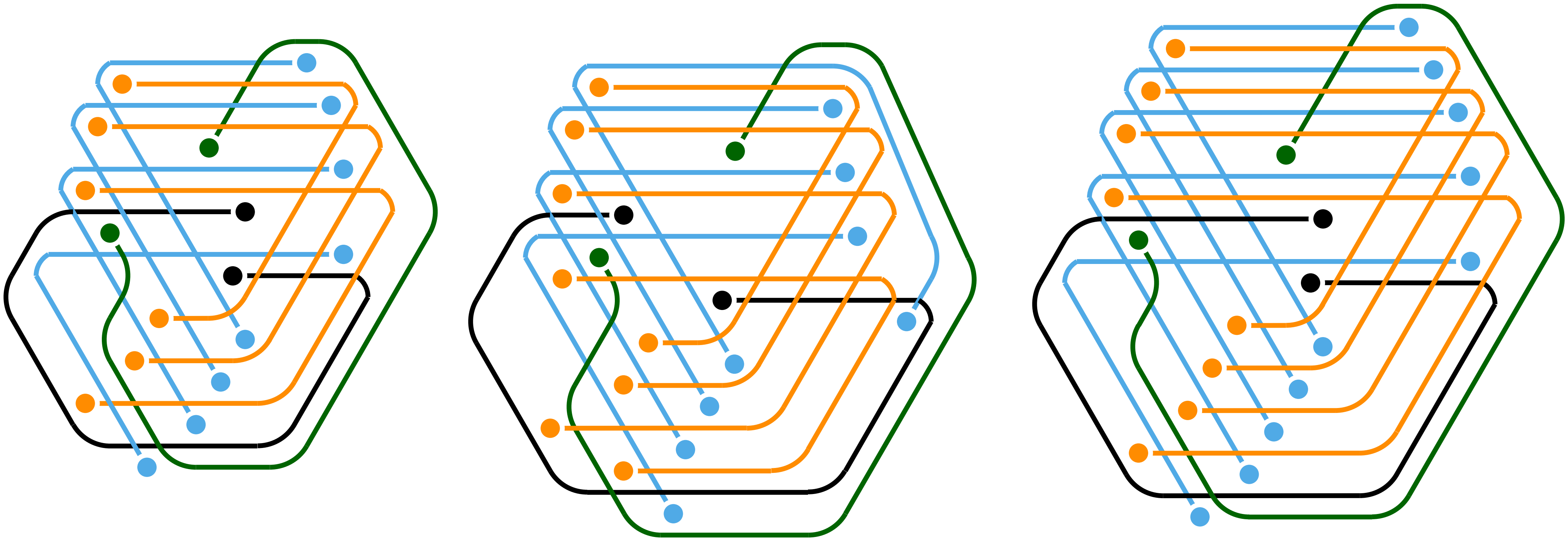
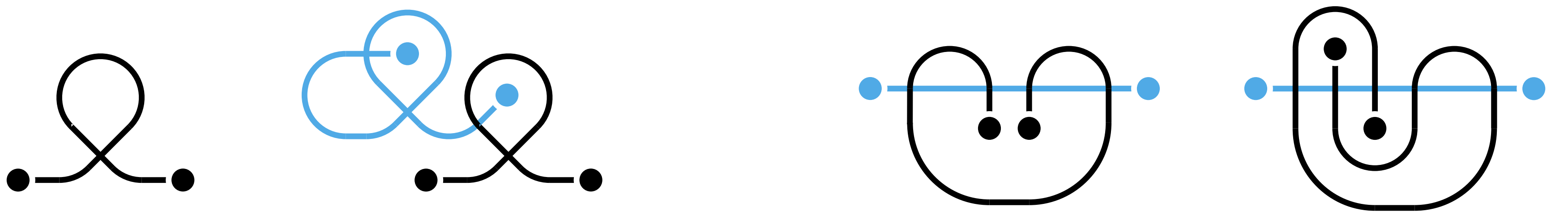


Only for $k \geq 6$

Conclusion

Tight bounds for k-planar drawings of multigraphs

For k-planar drawings of simple graphs less edges are possible



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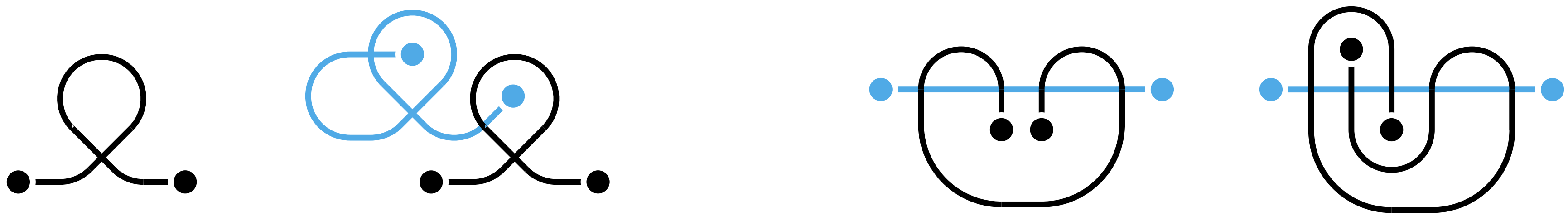


Future work:

- Tighten the bounds for simple graphs $\rightarrow$ stay tuned
- Study the setting for connected graphs, e.g. trees
- Do not fix the drawing

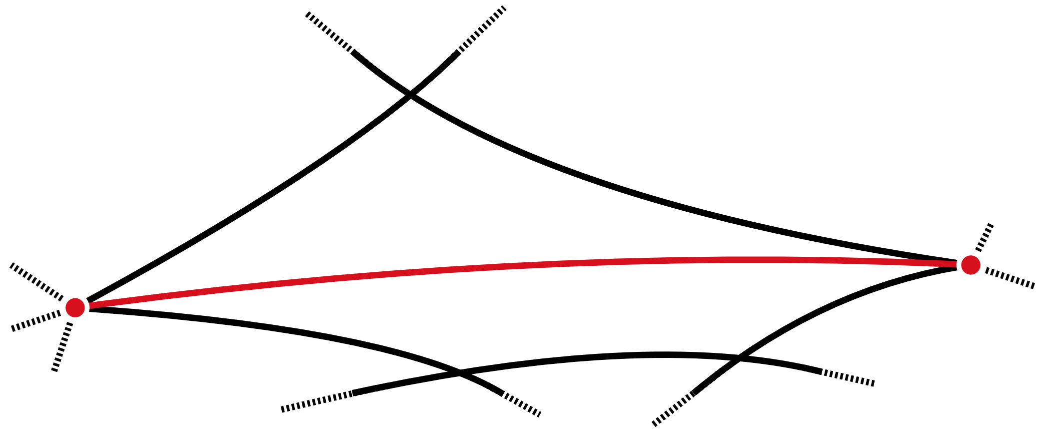
Thank you!

k	restrictions	minimum number of edges of saturated k-planar drawings on n vertices
$k \geq 4$	no restriction	$\frac{2}{k-(k \bmod 2)} \cdot (n-1)$
	I no incident crossings	
$k \geq 4$	**S** no selfcrossings	$\frac{2}{k-1} \cdot (n-1)$
	S no self- and **I** no incident crossings	
$k \geq 4$	**M** no multicrossings	$\frac{2(k-1)}{(k-1)(k-2)+2} \cdot (n-1)$
$k \geq 4$	**S** no self- and **M** no multicrossings	$\frac{2(k+1)}{k(k-1)} \cdot (n-1)$
$k = 4$	**I** no incident and **M** no multicrossings	$\frac{4}{5} \cdot (n-1)$
$k \geq 5$		$\frac{2(k-1)}{(k-1)(k-2)+2} \cdot (n-1)$
$k \geq 6$	**S** no self-, **M** no multi-, and **I** no incident crossings	$\frac{2(k+1)}{k(k-1)} \cdot (n-1)$
	S no self-, **M** no multi-, **I** no incident crossings, and **H** homotopy-free	

Filled and essentially-2-connected drawings

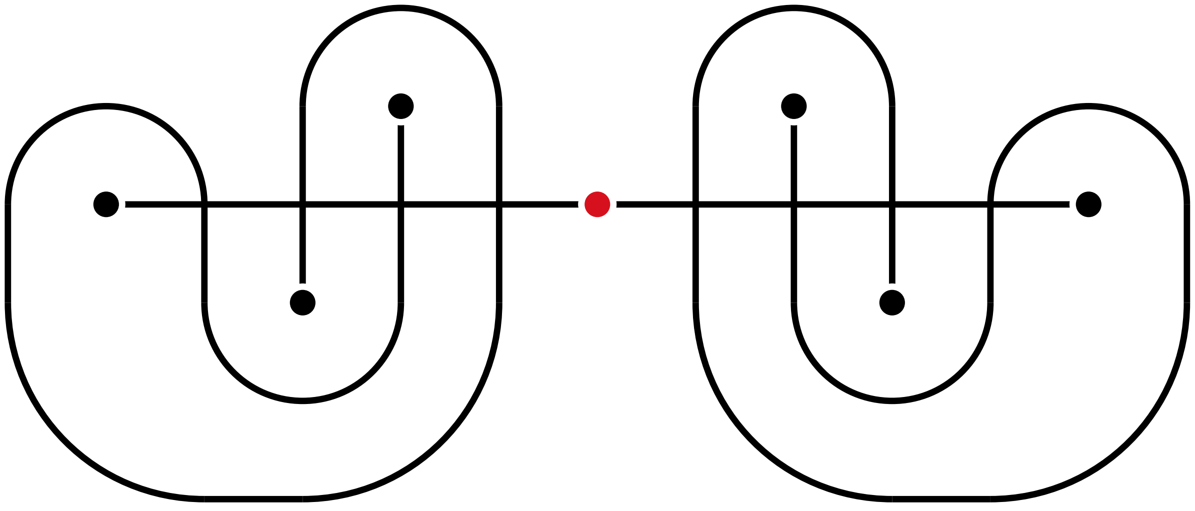
Filled drawings

Drawing D is filled if any two vertices on the boundary of a cell c in D are connected via an uncrossed edge in the boundary of c.

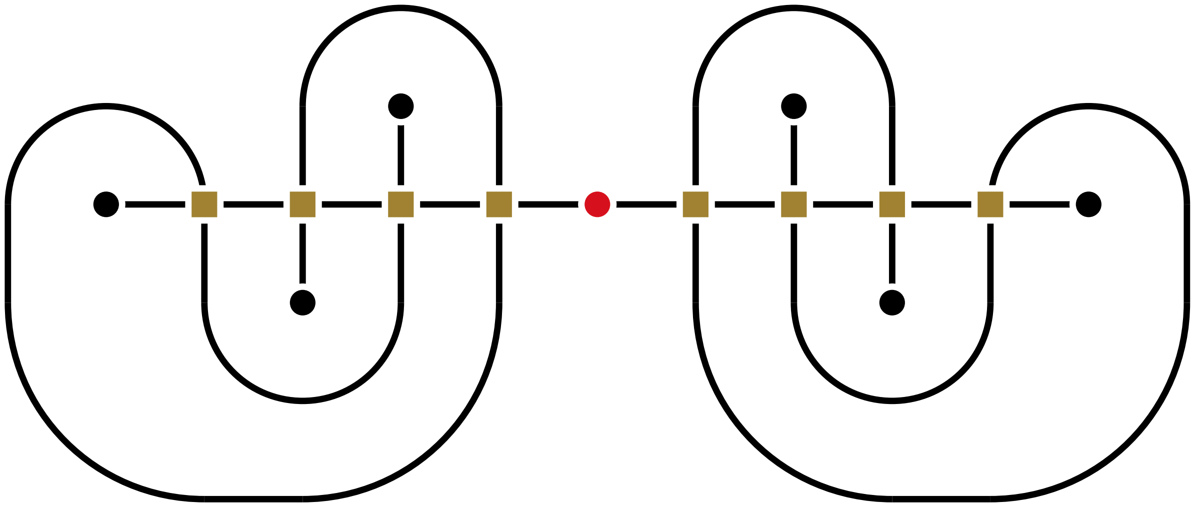


Let Γ be some drawing style from $\{\textbf{S}, \textbf{I}, \textbf{M}\}$ then every k-planar saturated drawing for drawing style Γ is filled.

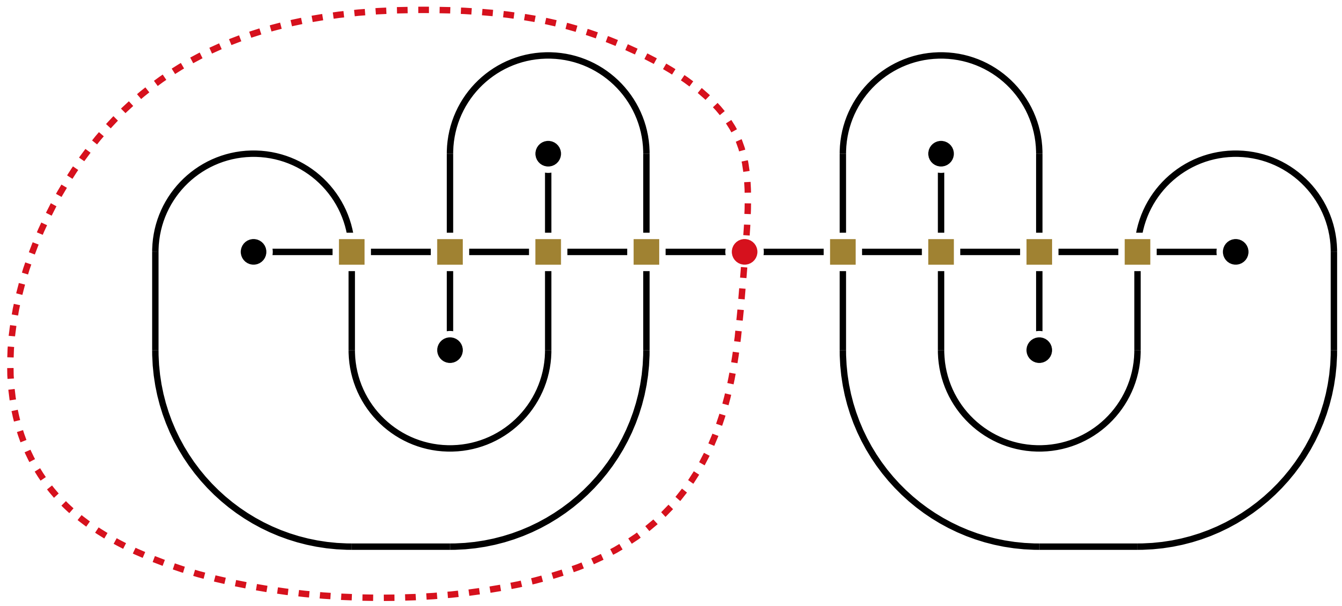
Essentially-2-connected



Essentially-2-connected

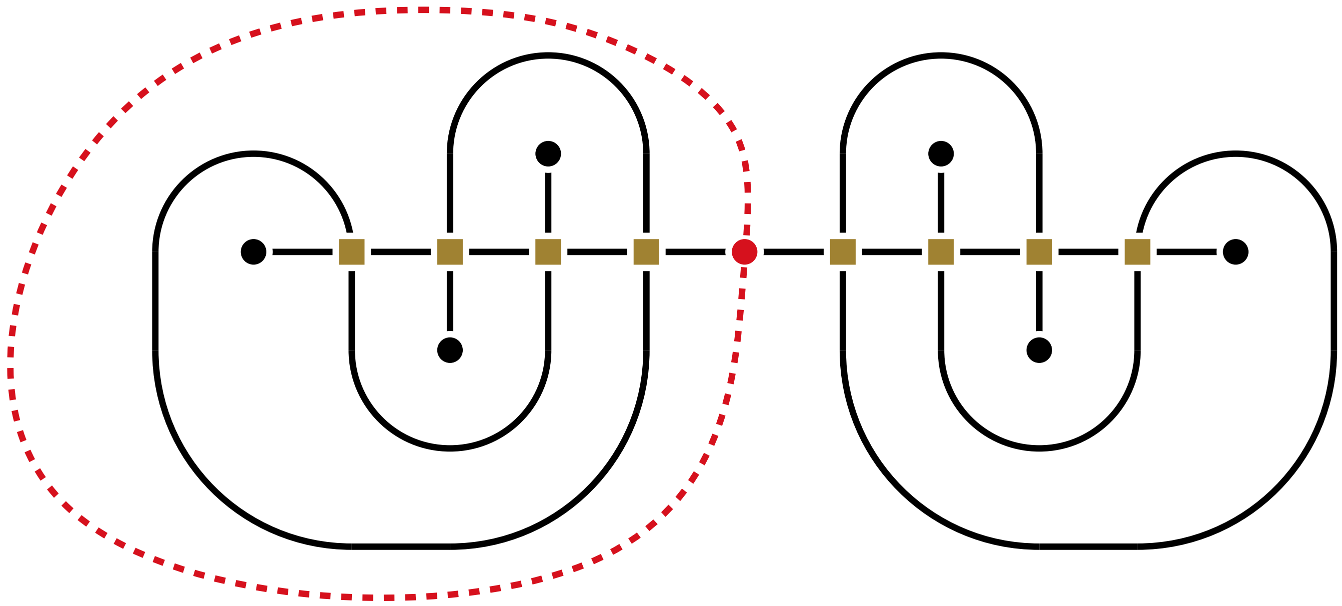


Essentially-2-connected

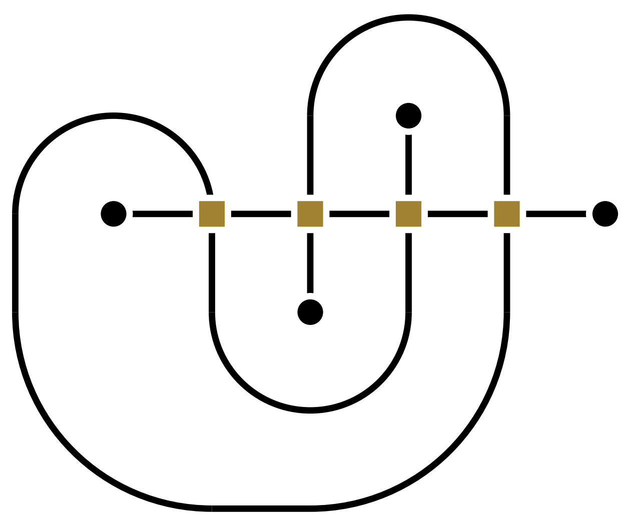


Not essentially-2-connected

Essentially-2-connected



Not essentially-2-connected



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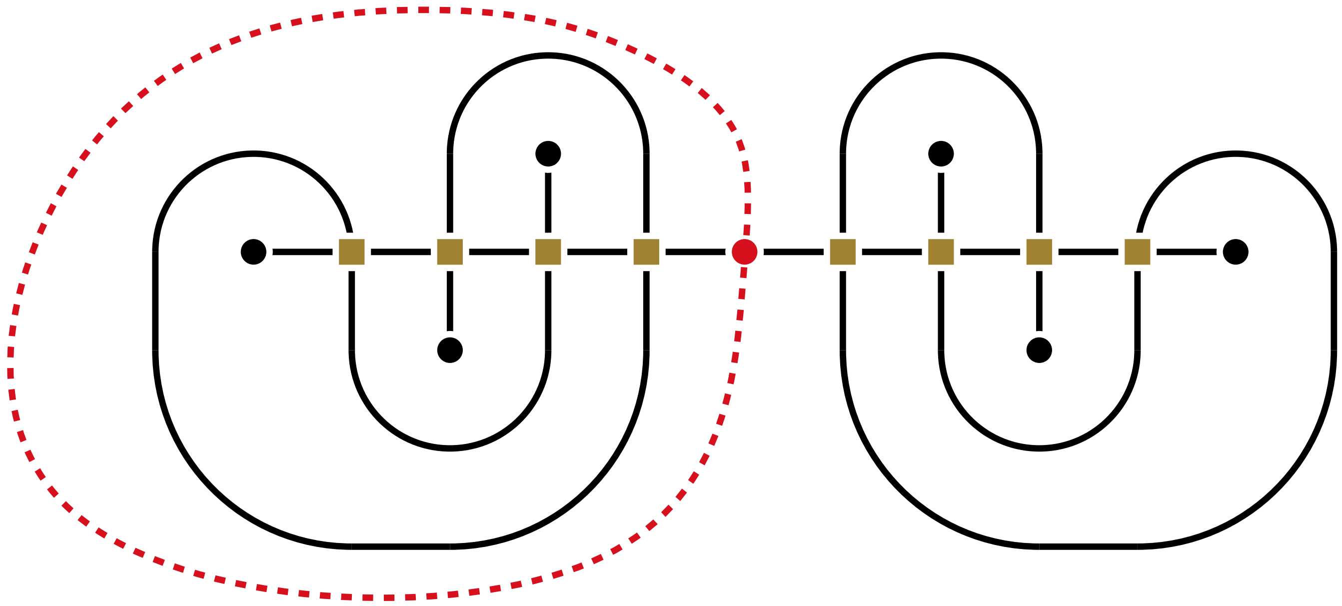

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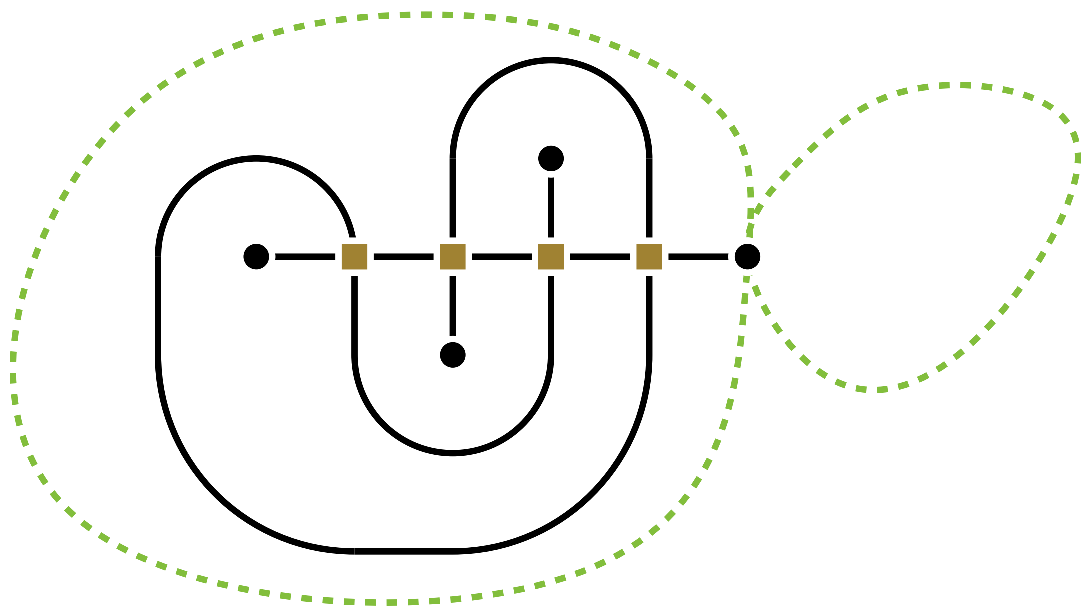

Inductive drawings

For every drawing style Γ from $\{\textbf{S}, \textbf{I}, \textbf{M}\}$ and every filled drawing $D \in \Gamma$ we have $m_D \geq \alpha_\Gamma \cdot (n_D + c_0(D) - 1)$ where

$$\alpha_\Gamma = \min \left\{ \frac{m_{D'}}{n_{D'} + c_0(D') - 1} : D' \in \Gamma \text{ is filled and e-2-c} \right\}.$$

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True when D is essentially-2-connected

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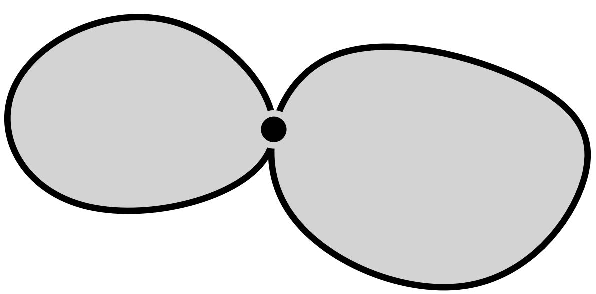
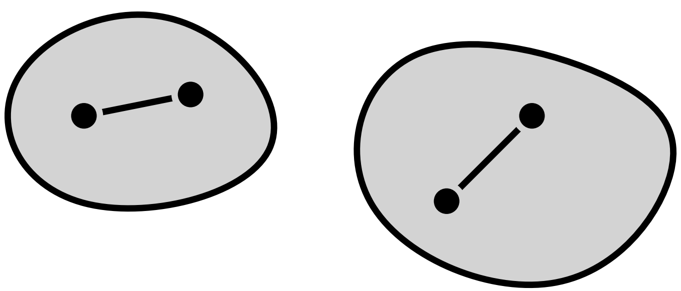


Cut-vertex



Two components

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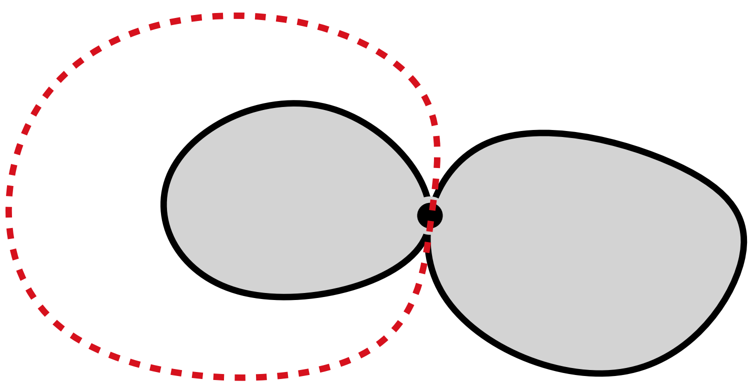

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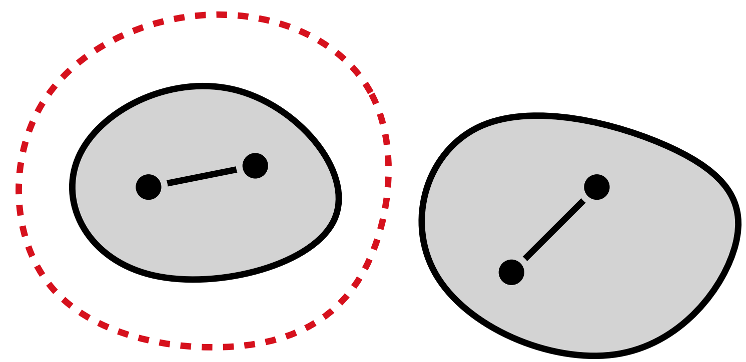

Two components

Exact bound

For every drawing style Γ from $\{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ and every filled drawing $D \in \Gamma$ we have $m_D \geq \alpha_\Gamma \cdot (n_D + c_0(D) - 1)$ where

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$$\frac{m_D}{n_D - 1} \geq \frac{m_D}{n_D + c_0(D) - 1} \geq \alpha_\Gamma = \min_{D'} \frac{m_{D'}}{n_{D'} + c_0(D') - 1}$$
$$= \min_{D'} \frac{2}{k-2} \cdot \frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1}$$

Lower bound

$$\frac{m_D}{n_D - 1} \geq \frac{m_D}{n_D + c_0(D) - 1} \geq \alpha = \min_{D'} \frac{m_{D'}}{n_{D'} + c_0(D') - 1}$$
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$$\alpha \geq \min_{D'} \frac{2}{k-2} \cdot \frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1} \geq \frac{2}{k-2} \cdot \frac{n_T - 2}{n_T - 1}$$

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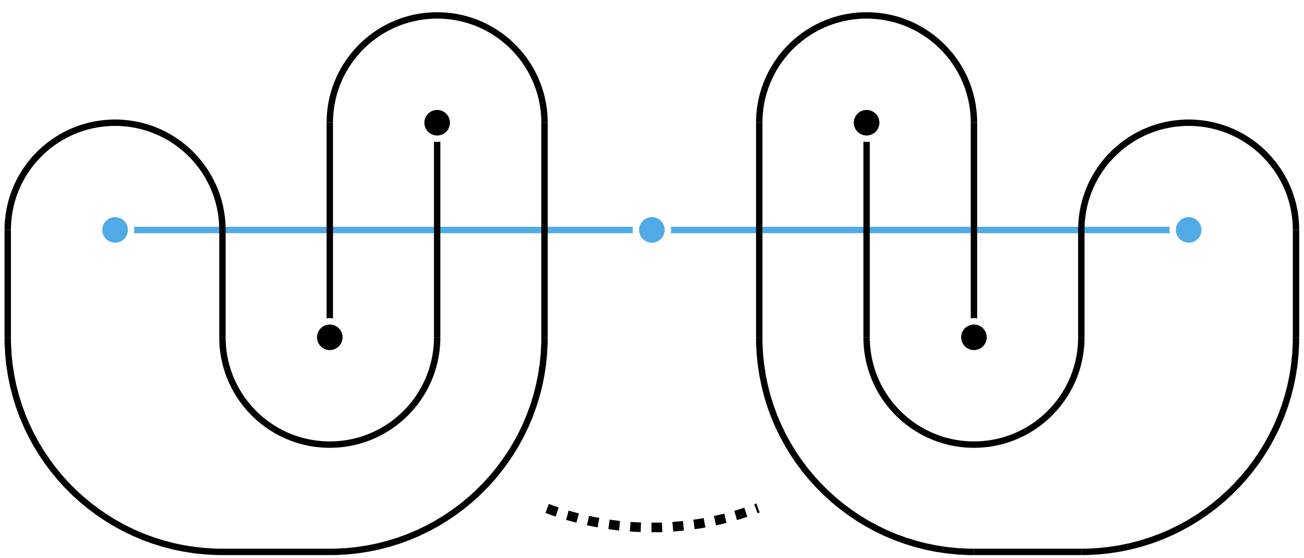

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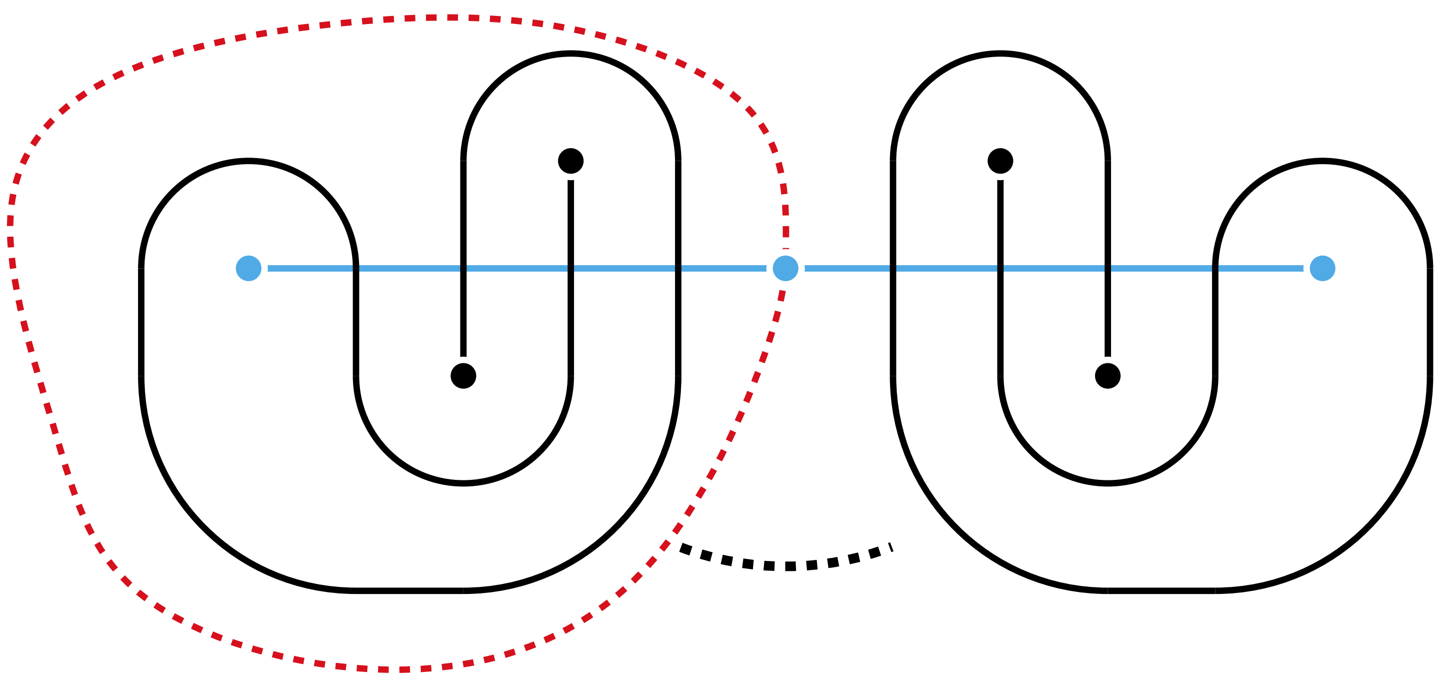

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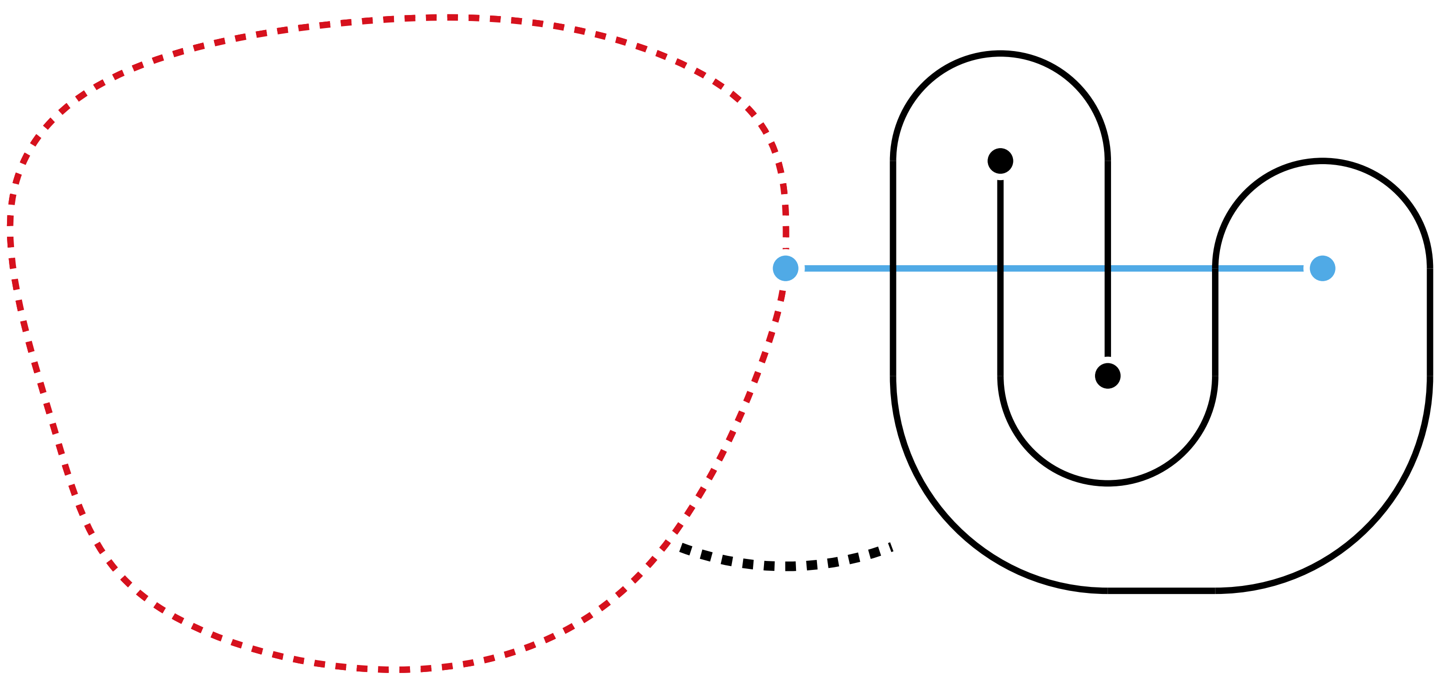

Simple graphs

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$\alpha \geq \frac{2}{k+2}$ best lower bound so far for all styles

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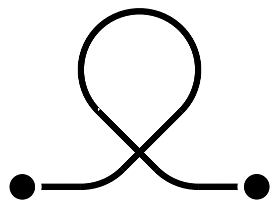
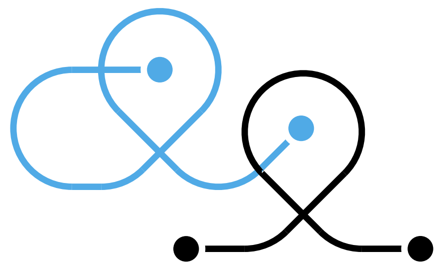
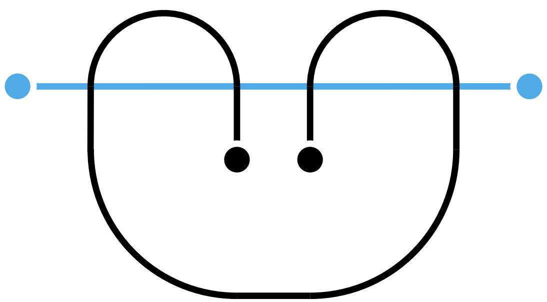
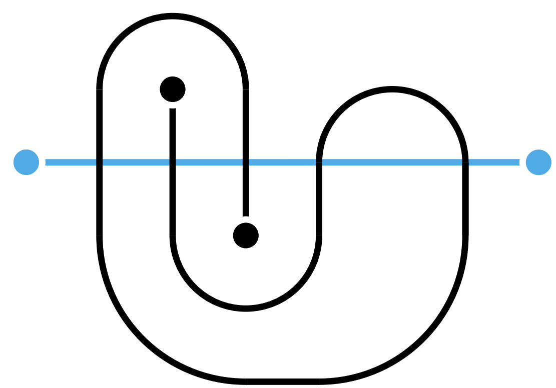




$$\alpha = \frac{2}{k+2} \text{ vs } \frac{2}{k-(k \bmod 2)}$$





$$\alpha = \frac{2}{k+((k+1) \bmod 2)} \text{ vs } \frac{2}{k-1}$$

Simple graphs

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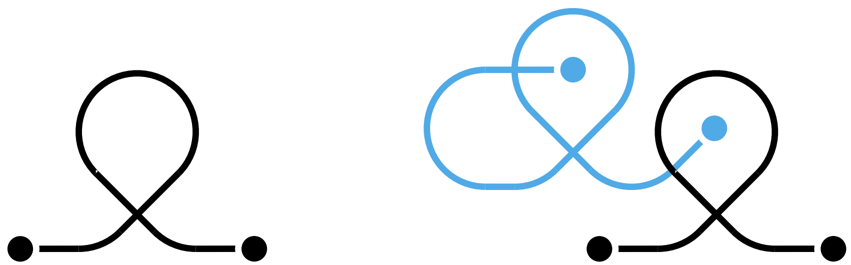

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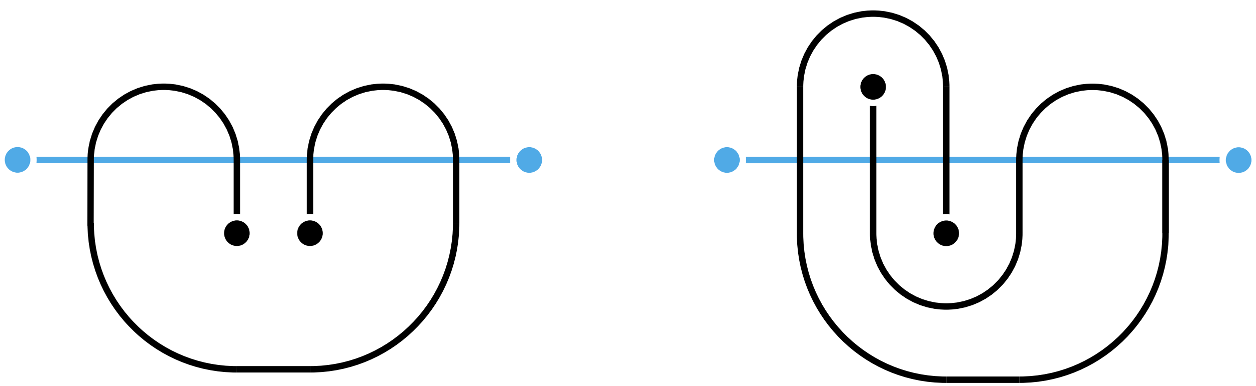
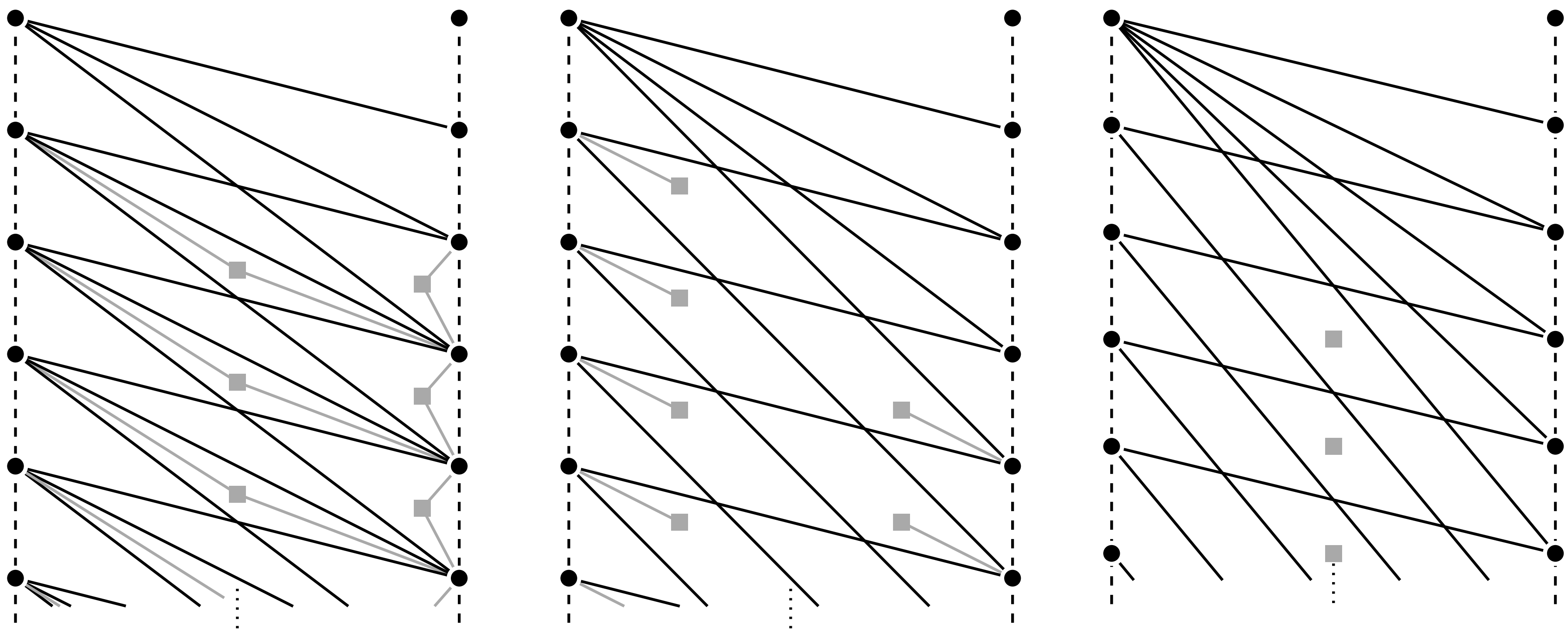


$$\alpha = \frac{2}{k+((k+1) \mod 2)} \text{ vs } \frac{2}{k-1}$$

$\alpha \leq \frac{2}{k-1}$ as soon as $\{\mathbf{M}\}$ in the style

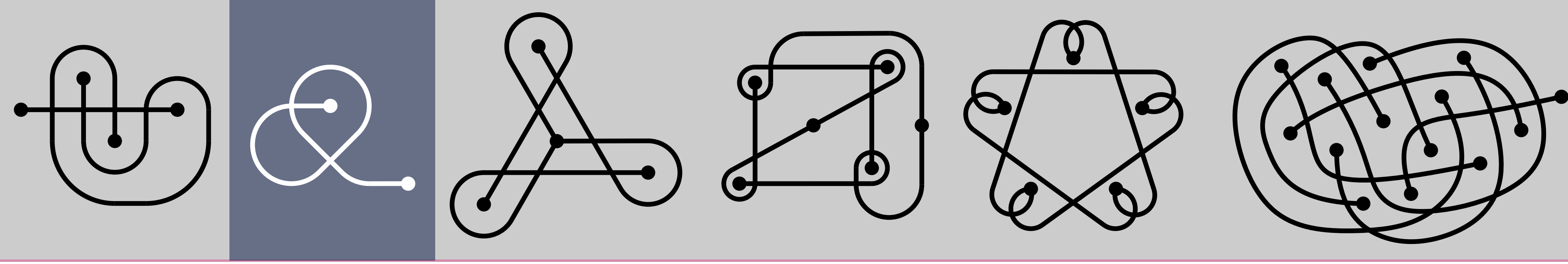
No multicrossings

Different type of construction for $\mathbf{M} \in \Gamma$



$k = 1$ $k = 2$ $k \geq 3$

[Brandenburg et al.'12] [Auer et al.'12]



We assume $1 \leq$ #crossed edges in $D \leq m_D < m_0$
Additionally: $\varepsilon(D)$ gives that

$$2 \cdot \#\text{crossings} \geq k \cdot \#\text{crossed edges} - 1$$

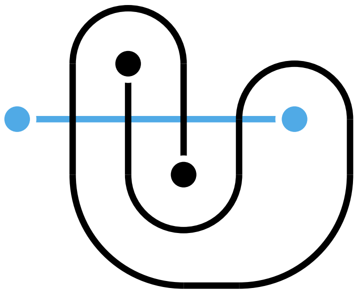
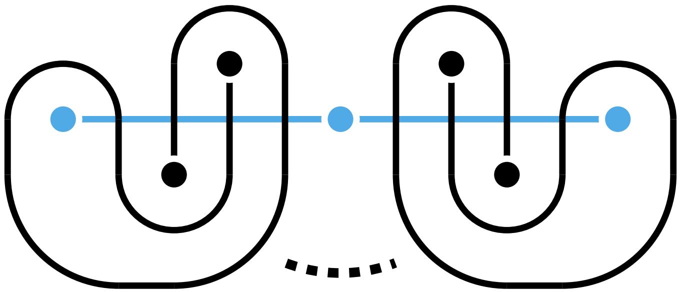
$\Rightarrow m_D = 1$ and this edge has exactly $\frac{k-1}{2}$ selfcrossings

Euler on planarization of $D \Rightarrow n_D + c_0(D) = (k+1)/2$

$$\frac{n_D + c_0(D) - 2 + \varepsilon(D)}{n_D + c_0(D') - 1} \geq \frac{(k+1)/2-2+1/2}{(k+1)/2-1}$$

$$= \frac{k-2}{k-1} = \frac{n_{D_T}-2}{n_{D_T}-1}.$$

Lower bound—Glimpse

 $\in \{\mathbf{S}\}$ and 4-planar 

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is true for any k-planar drawing D in the style that is **filled** and **essentially-2-connected**.

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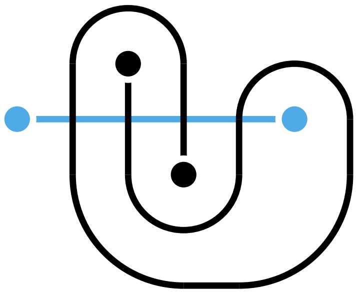
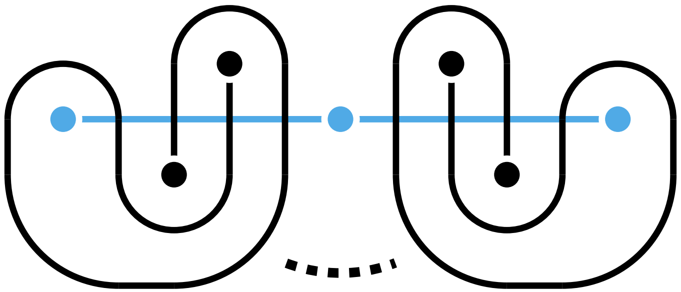
 $\in \{\mathbf{S}\}$ and 4-planar 

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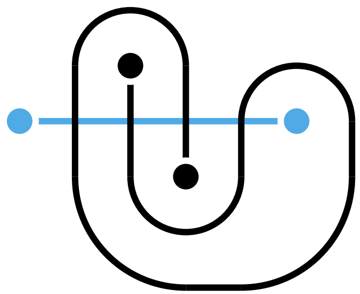
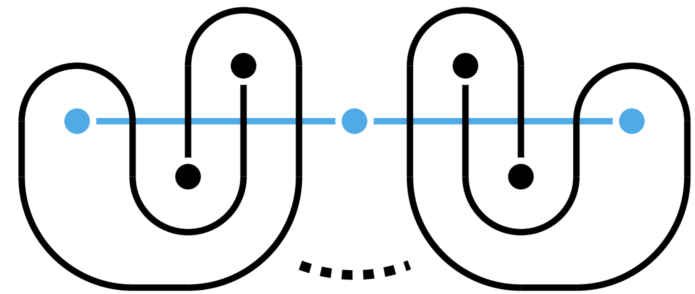
We can assume that:

- $0 \leq \varepsilon(D) < 1$
- $n_D + c_0(D) \leq n_T - 1$

Lower bound—Glimpse

 $\in \{\mathbf{S}\}$ and 4-planar



For the lower bound we have to check that

$$\frac{n_D + c_0(D) - 2 + \varepsilon(D)}{n_D + c_0(D) - 1} \geq \frac{n_T - 2}{n_T - 1}$$

is true for any k-planar drawing D in the style that is **filled** and **essentially-2-connected**.

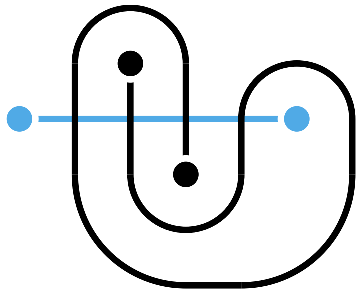
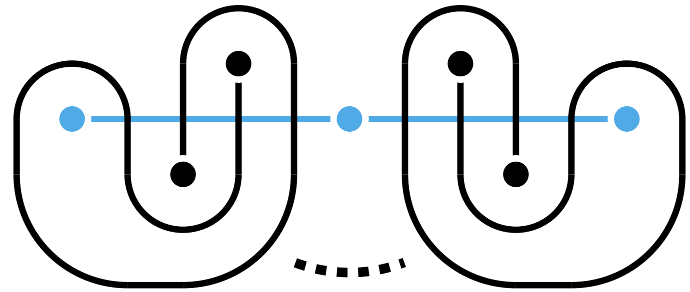
We can assume that:

- $0 \leq \varepsilon(D) < 1$
- $n_D + c_0(D) \leq n_T - 1$

$\Rightarrow$ the above equation holds unless:

$$m_D = \frac{2}{k-2}(n_D + c_0(D) - 2 + \varepsilon(D)) < \frac{2}{k-2}(n_T - 1 - 2 + 1) = m_T$$

Lower bound—Glimpse

 $\in \{\mathbf{S}\}$ and 4-planar 

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is true for any k-planar drawing D in the style that is **filled** and **essentially-2-connected**.

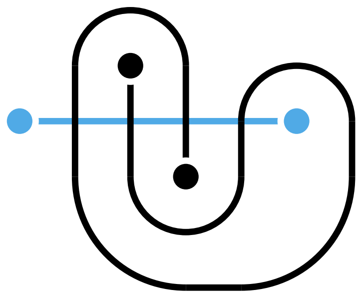
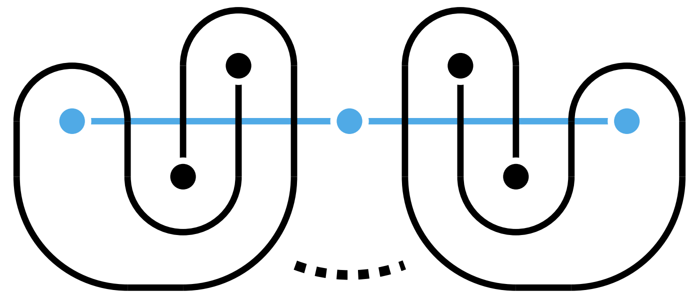
We can assume that:

- $0 \leq \varepsilon(D) < 1$
- $n_D + c_0(D) \leq n_T - 1$

$\Rightarrow$ the above equation holds unless:

$$m_D < m_T$$

Lower bound—Glimpse

 $\in \{\mathbf{S}\}$ and 4-planar 

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is true for any k-planar drawing D in the style that is **filled** and **essentially-2-connected**.

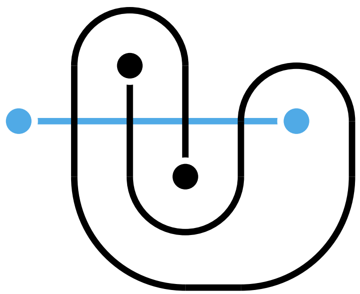
We can assume that:

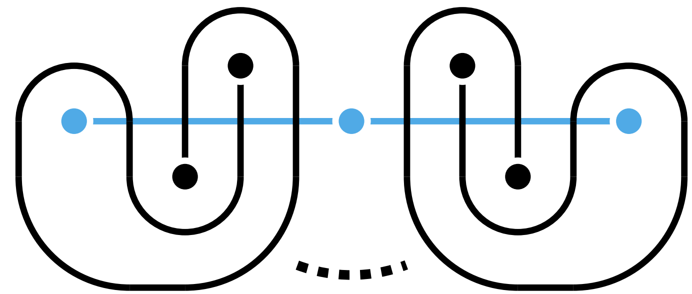
- $0 \leq \varepsilon(D) < 1$
- $n_D + c_0(D) \leq n_T - 1$

$\Rightarrow$ the above equation holds unless:

$$1 \leq \#\text{crossed edges} \leq m_D < m_T = 2$$

Lower bound—Glimpse

 $\in \{\mathbf{S}\}$ and 4-planar



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